

第三章 不定积分

§1. 最简单的不定积分

1° 不定积分的概念 若 $f(x)$ 为连续函数及 $F'(x) = f(x)$, 则

$$\int f(x) dx = F(x) + C,$$

式中 C 为任意常数.

2° 不定积分的基本性质:

$$(a) d\left[\int f(x) dx\right] = f(x) dx; \quad (b) \int d\Phi(x) = \Phi(x) + C;$$

$$(c) \int Af(x) dx = A \int f(x) dx \quad (A = \text{常数});$$

$$(d) \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx.$$

3° 最简积分表:

$$I. \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1);$$

$$II. \int \frac{dx}{x} = \ln|x| + C \quad (x \neq 0);$$

$$III. \int \frac{dx}{1+x^2} = \begin{cases} \operatorname{arctg}x + C, \\ -\operatorname{arccotg}x + C, \end{cases}$$

$$\text{IV. } \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C;$$

$$\text{V. } \int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C; \end{cases}$$

$$\text{VI. } \int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln |x + \sqrt{x^2 \pm 1}| + C;$$

$$\text{VII. } \int a^x dx = \frac{a^x}{\ln a} + C (a > 0, a \neq 1); \int e^x dx = e^x + C;$$

$$\text{VIII. } \int \sin x dx = -\cos x + C;$$

$$\text{IX. } \int \cos x dx = \sin x + C;$$

$$\text{X. } \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C;$$

$$\text{XI. } \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C;$$

$$\text{XII. } \int \operatorname{sh} x dx = \operatorname{ch} x + C;$$

$$\text{XIII. } \int \operatorname{ch} x dx = \operatorname{sh} x + C;$$

$$\text{XIV. } \int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C;$$

$$\text{XV. } \int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C.$$

4° 积分的基本方法

(a) 引入新变数法 若

$$\int f(x) dx = F(x) + C,$$

则 $\int f(u) du = F(u) + C$, 式中 $u = \varphi(x)$.

(6) 分项积分法 若

$$f(x) = f_1(x) + f_2(x),$$

则 $\int f(x) dx = \int f_1(x) dx + \int f_2(x) dx$.

(B) 代入法 假设

$x = \varphi(t)$, 式中 $\varphi(t)$ 及其导函数 $\varphi'(t)$ 为连续的,

则得 $\int f(x) dx = \int f[\varphi(t)] \varphi'(t) dt$.

(r) 分部积分法 若 u 和 v 为 x 的可微分函数,

则 $\int u dv = uv - \int v du$.

利用最简积分表, 求出下列积分*:

1628. $\int (3-x^2)^3 dx$.

* 本章在叙述习题及其解答过程中, 凡出现的函数, 无论是被积函数还是原函数, 均默认是在有意义的定义域上进行的, 例如最简积分表中 I 里当 $n \leq -2$ 时, 要求 $x \neq 0$; IV 中要求 $|x| \neq 1$; V 中要求 $|x| < 1$; 以及 VI 中, 当取负号时要求 $|x| > 1$; 等等, 就未加声明, 在题解中也有相当多的类似情况。因此, 如无特别声明, 在一般情形下, 这些定义域是很容易被读者确定的, 此处就不再予以一一指明。——题解编者注。

$$\begin{aligned} \text{解 } \int (3-x^2)^3 dx &= \int (27-27x^2+9x^4-x^6) dx \\ &= 27x-9x^3+\frac{9}{5}x^5-\frac{1}{7}x^7+C. \end{aligned}$$

$$1629. \int x^2(5-x)^4 dx.$$

$$\begin{aligned} \text{解 } \int x^2(5-x)^4 dx &= \int (625x^2-500x^3+150x^4-20x^5+x^6) dx \\ &= \frac{625}{3}x^3-125x^4+30x^5-\frac{10}{3}x^6+\frac{1}{7}x^7+C. \end{aligned}$$

$$1630. \int (1-x)(1-2x)(1-3x) dx.$$

$$\begin{aligned} \text{解 } \int (1-x)(1-2x)(1-3x) dx &= \int (1-6x+11x^2-6x^3) dx \\ &= x-3x^2+\frac{11}{3}x^3-\frac{3}{2}x^4+C. \end{aligned}$$

$$1631. \int \left(\frac{1-x}{x}\right)^2 dx.$$

$$\begin{aligned} \text{解 } \int \left(\frac{1-x}{x}\right)^2 dx &= \int \left(\frac{1}{x^2}-\frac{2}{x}+1\right) dx \\ &= -\frac{1}{x}-2\ln|x|+x+C. \end{aligned}$$

$$1632. \int \left(\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}\right) dx.$$

$$\text{解 } \int \left(\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}\right) dx = a\ln|x|-\frac{a^2}{x}-\frac{a^3}{2x^2}+C.$$

$$1633. \int \frac{x+1}{\sqrt{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x+1}{\sqrt{x}} dx &= \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx \\ &= \frac{2}{3} x\sqrt{x} + 2\sqrt{x} + C. \end{aligned}$$

$$1634. \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx &= \int (x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}}) dx \\ &= \frac{4}{5} x\sqrt[4]{x} - \frac{24}{17} x\sqrt[2]{x^5} + \frac{4}{3} \sqrt[4]{x^3} + C. \end{aligned}$$

$$1635. \int \frac{(1-x)^3}{x\sqrt[3]{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{(1-x)^3}{x\sqrt[3]{x}} dx &= \int (x^{-\frac{4}{3}} - 3x^{-\frac{1}{3}} + 3x^{\frac{2}{3}} - x^{\frac{5}{3}}) dx \\ &= -\frac{3}{\sqrt[3]{x}} \left(1 + \frac{8}{2}x - \frac{3}{5}x^2 + \frac{1}{8}x^3 \right) + C. \end{aligned}$$

$$1636. \int \left(1 - \frac{1}{x^2} \right) \sqrt{x\sqrt{x}} dx.$$

$$\text{解} \quad \int \left(1 - \frac{1}{x^2} \right) \sqrt{x\sqrt{x}} dx = \int (x^{\frac{3}{4}} - x^{-\frac{5}{4}}) dx$$

$$= \frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C = \frac{4(x^2+7)}{7\sqrt[4]{x}} + C.$$

$$1637. \int \frac{(\sqrt{2x} - \sqrt[3]{3x})^2}{x} dx.$$

$$\text{解} \int \frac{(\sqrt{2x} - \sqrt[3]{3x})^2}{x} dx$$

$$= \int (2 - 2\sqrt[6]{72}x^{-\frac{1}{6}} + \sqrt[3]{9}x^{-\frac{1}{3}}) dx$$

$$= 2x - \frac{12}{5}\sqrt[6]{72}x^{\frac{5}{6}} + \frac{3}{2}\sqrt[3]{9}x^{\frac{2}{3}} + C.$$

$$1638. \int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx.$$

$$\text{解} \int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx = \int \frac{x^2 + \frac{1}{x^2}}{x^3} dx$$

$$= \int \left(\frac{1}{x} + \frac{1}{x^5} \right) dx = \ln|x| - \frac{1}{4x^4} + C.$$

$$1639. \int \frac{x^2}{1+x^2} dx.$$

$$\text{解} \int \frac{x^2}{1+x^2} dx = \int \left(1 - \frac{1}{x^2+1} \right) dx$$

$$= x - \text{arctg}x + C.$$

$$1640. \int \frac{x^2}{1-x^2} dx.$$

$$\text{解} \int \frac{x^2}{1-x^2} dx = \int \left(-1 + \frac{1}{1-x^2} \right) dx$$

$$= -x + \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C.$$

1641. $\int \frac{x^2+3}{x^2-1} dx.$

解 $\int \frac{x^2+3}{x^2-1} dx = \int \left(1 + \frac{4}{x^2-1} \right) dx$

$$= x + 2 \ln \left| \frac{x-1}{x+1} \right| + C.$$

1642. $\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx.$

解 $\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx$

$$= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}} \right) dx$$

$$= \arcsin x + \ln(x + \sqrt{1+x^2}) + C.$$

1643. $\int \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^4-1}} dx.$

解 $\int \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^4-1}} dx$

$$= \int \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2+1}} \right) dx$$

$$= \ln \left| \frac{x + \sqrt{x^2-1}}{x + \sqrt{x^2+1}} \right| + C.$$

$$1644. \int (2^x + 3^x)^2 dx.$$

$$\begin{aligned} \text{解} \quad \int (2^x + 3^x)^2 dx &= \int (4^x + 2 \cdot 6^x + 9^x) dx \\ &= \frac{4^x}{\ln 4} + 2 \cdot \frac{6^x}{\ln 6} + \frac{9^x}{\ln 9} + C. \end{aligned}$$

$$1645. \int \frac{2^{x+1} - 5^{x-1}}{10^x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{2^{x+1} - 5^{x-1}}{10^x} dx &= \int \left[2 \left(\frac{1}{5} \right)^x - \frac{1}{5} \left(\frac{1}{2} \right)^x \right] dx \\ &= -\frac{2}{\ln 5} \left(\frac{1}{5} \right)^x + \frac{1}{5 \ln 2} \left(\frac{1}{2} \right)^x + C. \end{aligned}$$

$$1646. \int \frac{e^{2x} + 1}{e^x + 1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{e^{2x} + 1}{e^x + 1} dx &= \int (e^{2x} - e^x + 1) dx \\ &= \frac{1}{2} e^{2x} - e^x + x + C. \end{aligned}$$

$$1647. \int (1 + \sin x + \cos x) dx.$$

$$\text{解} \quad \int (1 + \sin x + \cos x) dx = x - \cos x + \sin x + C.$$

$$1648. \int \sqrt{1 - \sin 2x} dx.$$

$$\begin{aligned}
 \text{解} \quad \int \sqrt{1 - \sin^2 x} dx &= \int \sqrt{(\cos x - \sin x)^2} dx \\
 &= \int (\operatorname{sgn}(\cos x - \sin x)) (\cos x - \sin x) dx \\
 &= (\sin x + \cos x) \cdot \operatorname{sgn}(\cos x - \sin x) + C.
 \end{aligned}$$

1649. $\int \operatorname{ctg}^2 x dx.$

$$\text{解} \quad \int \operatorname{ctg}^2 x dx = \int (\operatorname{csc}^2 x - 1) dx = -\operatorname{ctg} x - x + C.$$

1650. $\int \operatorname{tg}^2 x dx.$

$$\text{解} \quad \int \operatorname{tg}^2 x dx = \int (\sec^2 x - 1) dx = \operatorname{tg} x - x + C.$$

1651. $\int (a \operatorname{sh} x + b \operatorname{ch} x) dx.$

$$\text{解} \quad \int (a \operatorname{sh} x + b \operatorname{ch} x) dx = a \operatorname{ch} x + b \operatorname{sh} x + C.$$

1652. $\int \operatorname{th}^2 x dx.$

$$\text{解} \quad \int \operatorname{th}^2 x dx = \int \left(1 - \frac{1}{\operatorname{ch}^2 x} \right) dx = x - \operatorname{th} x + C.$$

1653. $\int \operatorname{cth}^2 x dx.$

$$\text{解} \quad \int \operatorname{cth}^2 x dx = \int \left(1 + \frac{1}{\operatorname{sh}^2 x} \right) dx = x - \operatorname{cth} x + C.$$

1654. 证明: 若

$$\int j(x) dx = F(x) + C,$$

则

$$\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C \quad (a \neq 0).$$

证 由 $\int f(x)dx = F(x) + C$ 得知 $F'(x) = f(x)$, 因而
 有 $F'(ax+b) = f(ax+b)$, 且 $\frac{d}{dx} \left[\frac{1}{a} F(ax+b) \right]$
 $= F'(ax+b)$, 于是

$$\frac{d}{dx} \left[\frac{1}{a} F(ax+b) \right] = f(ax+b),$$

所以

$$\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C.$$

求出下列积分:

1655. $\int \frac{dx}{x+a}.$

解 $\int \frac{dx}{x+a} = \ln|x+a| + C.$

1656. $\int (2x-3)^{10} dx.$

解 $\int (2x-3)^{10} dx = \frac{1}{2} \cdot \frac{1}{11} (2x-3)^{11} + C$
 $= \frac{1}{22} (2x-3)^{11} + C.$

1657. $\int \sqrt[3]{1-3x} dx.$

解 $\int \sqrt[3]{1-3x} dx = -\frac{1}{3} \cdot \frac{3}{4} (1-3x)^{\frac{4}{3}} + C$
 $= -\frac{1}{4} (1-3x)^{\frac{4}{3}} + C.$

$$1658. \int \frac{dx}{\sqrt{2-5x}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{2-5x}} &= -\frac{1}{5} \cdot 2(2-5x)^{\frac{1}{2}} + C \\ &= -\frac{2}{5} \sqrt{2-5x} + C. \end{aligned}$$

$$1659. \int \frac{dx}{(5x-2)^{\frac{3}{2}}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{(5x-2)^{\frac{3}{2}}} &= \frac{1}{5} \cdot \left(-\frac{2}{3}\right) (5x-2)^{-\frac{3}{2}} + C \\ &= -\frac{2}{15(5x-2)^{\frac{3}{2}}} + C. \end{aligned}$$

$$1660^{+*}. \int \frac{\sqrt[5]{1-2x+x^2}}{1-x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sqrt[5]{1-2x+x^2}}{1-x} dx &= \int (1-x)^{-\frac{3}{5}} dx \\ &= -\frac{5}{2} \sqrt[5]{(1-x)^2} + C. \end{aligned}$$

* 题号右上角带“+”号表示题解答案与原习题集中译本所附答案不一致，以后不再说明。中译本基本是按俄文第二版翻译的。俄文第二版中有一些错误已在俄文第三版中改正。

$$1661. \int \frac{dx}{2+3x^2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{2+3x^2} &= \int \frac{dx}{(\sqrt{2})^2 + (\sqrt{\frac{3}{2}}x)^2} \\ &= \frac{1}{\sqrt{6}} \operatorname{arctg} \left(x\sqrt{\frac{3}{2}} \right) + C. \end{aligned}$$

$$1662. \int \frac{dx}{2-3x^2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{2-3x^2} &= \frac{1}{2} \int \frac{dx}{1 - \left(\sqrt{\frac{3}{2}}x\right)^2} \\ &= \frac{1}{2} \cdot \sqrt{\frac{2}{3}} \cdot \frac{1}{2} \ln \left| \frac{1 + \sqrt{\frac{3}{2}}x}{1 - \sqrt{\frac{3}{2}}x} \right| + C \\ &= \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2} + x\sqrt{3}}{\sqrt{2} - x\sqrt{3}} \right| + C. \end{aligned}$$

$$1663. \int \frac{dx}{\sqrt{2-3x^2}}.$$

$$\text{解} \quad \int \frac{dx}{\sqrt{2-3x^2}} = \frac{1}{\sqrt{3}} \operatorname{arcsin} \left(x\sqrt{\frac{3}{2}} \right) + C.$$

$$1664. \int \frac{dx}{\sqrt{3x^2-2}}.$$

$$\text{解} \quad \int \frac{dx}{\sqrt{3x^2-2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{3}{2}}x\right)^2 - 1}}$$

$$= \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2}{3}} \ln \left| x \sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}x^2 - 1} \right| + C_1$$

$$= \frac{1}{\sqrt{3}} \ln |x\sqrt{3} + \sqrt{3x^2 - 2}| + C.$$

1665. $\int (e^{-x} + e^{-2x}) dx.$

解 $\int (e^{-x} + e^{-2x}) dx = -(e^{-x} + \frac{1}{2}e^{-2x}) + C.$

1666. $\int (\sin 5x - \sin 5a) dx.$

解 $\int (\sin 5x - \sin 5a) dx = -\frac{1}{5} \cos 5x - x \sin 5a + C.$

1667. $\int \frac{dx}{\sin^2(2x + \frac{\pi}{4})}.$

解 $\int \frac{dx}{\sin^2(2x + \frac{\pi}{4})} = -\frac{1}{2} \operatorname{ctg}(2x + \frac{\pi}{4}) + C.$

1668. $\int \frac{dx}{1 + \cos x}.$

解 $\int \frac{dx}{1 + \cos x} = \frac{1}{2} \int \frac{dx}{\cos^2 \frac{x}{2}} = \operatorname{tg} \frac{x}{2} + C.$

1669. $\int \frac{dx}{1 - \cos x}.$

解 $\int \frac{dx}{1 - \cos x} = \frac{1}{2} \int \frac{dx}{\sin^2 \frac{x}{2}} = -\operatorname{ctg} \frac{x}{2} + C.$

$$1670. \int \frac{dx}{1 + \sin x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{1 + \sin x} &= \int \frac{dx}{1 + \cos\left(\frac{\pi}{2} - x\right)} \\ &= -\operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right) + C. \end{aligned}$$

$$1671. \int [\operatorname{sh}(2x+1) + \operatorname{ch}(2x-1)] dx.$$

$$\begin{aligned} \text{解} \quad \int [\operatorname{sh}(2x+1) + \operatorname{ch}(2x-1)] dx \\ = \frac{1}{2} [\operatorname{ch}(2x+1) + \operatorname{sh}(2x-1)] + C. \end{aligned}$$

$$1672. \int \frac{dx}{\operatorname{ch}^2 \frac{x}{2}}.$$

$$\text{解} \quad \int \frac{dx}{\operatorname{ch}^2 \frac{x}{2}} = 2 \operatorname{th} \frac{x}{2} + C.$$

$$1673. \int \frac{dx}{\operatorname{sh}^2 \frac{x}{2}}.$$

$$\text{解} \quad \int \frac{dx}{\operatorname{sh}^2 \frac{x}{2}} = -2 \operatorname{cth} \frac{x}{2} + C.$$

用适当地变换被积函数的方法来求下列积分:

$$1674. \int \frac{x dx}{\sqrt{1-x^2}}.$$

$$\text{解 } \int \frac{x dx}{\sqrt{1-x^2}} = -\int \frac{d(1-x^2)}{2\sqrt{1-x^2}} = -\sqrt{1-x^2} + C.$$

$$1675. \int x^2 \sqrt[3]{1+x^3} dx.$$

$$\begin{aligned} \text{解 } \int x^2 \sqrt[3]{1+x^3} dx &= \frac{1}{3} \int (1+x^3)^{\frac{1}{3}} d(1+x^3) \\ &= \frac{1}{4} (1+x^3)^{\frac{4}{3}} + C. \end{aligned}$$

$$1676. \int \frac{x dx}{3-2x^2}.$$

$$\begin{aligned} \text{解 } \int \frac{x dx}{3-2x^2} &= -\frac{1}{4} \int \frac{d(3-2x^2)}{3-2x^2} \\ &= -\frac{1}{4} \ln|3-2x^2| + C. \end{aligned}$$

$$1677. \int \frac{x dx}{(1+x^2)^2}.$$

$$\begin{aligned} \text{解 } \int \frac{x dx}{(1+x^2)^2} &= \frac{1}{2} \int \frac{d(1+x^2)}{(1+x^2)^2} \\ &= -\frac{1}{2(1+x^2)} + C. \end{aligned}$$

$$1678. \int \frac{x dx}{4+x^4}.$$

$$\text{解 } \int \frac{x dx}{4+x^4} = \frac{1}{2} \int \frac{d(x^2)}{2^2+(x^2)^2} = \frac{1}{4} \operatorname{arc} \operatorname{tg} \frac{x^2}{2} + C.$$

$$1679. \int \frac{x^3 dx}{x^8-2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3 dx}{x^8 - 2} &= \frac{1}{8} \int \frac{d(x^4)}{(x^4)^2 - (\sqrt{2})^2} \\ &= \frac{1}{8\sqrt{2}} \ln \left| \frac{x^4 - \sqrt{2}}{x^4 + \sqrt{2}} \right| + C. \end{aligned}$$

$$1680. \int \frac{dx}{\sqrt{x}(1+x)}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{x}(1+x)} &= 2 \int \frac{d(\sqrt{x})}{1 + (\sqrt{x})^2} \\ &= 2 \operatorname{arctg} \sqrt{x} + C. \end{aligned}$$

$$1681. \int \sin \frac{1}{x} \cdot \frac{dx}{x^2}.$$

$$\text{解} \quad \int \sin \frac{1}{x} \cdot \frac{dx}{x^2} = - \int \sin \frac{1}{x} d\left(\frac{1}{x}\right) = \cos \frac{1}{x} + C.$$

$$1682. \int \frac{dx}{x\sqrt{x^2+1}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x\sqrt{x^2+1}} &= \int \frac{dx}{x|x|\sqrt{1+\frac{1}{x^2}}} \\ &= - \int \frac{d\left(\frac{1}{|x|}\right)}{\sqrt{1+\left(\frac{1}{|x|}\right)^2}} = - \ln \left(\frac{1}{|x|} + \sqrt{1+\frac{1}{x^2}} \right) + C \\ &= - \ln \left| \frac{1 + \sqrt{x^2+1}}{x} \right| + C. \end{aligned}$$

$$1683. \int \frac{dx}{x\sqrt{x^2-1}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{dx}{x\sqrt{x^2-1}} &= \int \frac{dx}{x|x|\sqrt{1-\frac{1}{x^2}}} \\
 &= -\int \frac{d\left(\frac{1}{|x|}\right)}{\sqrt{1-\left(\frac{1}{|x|}\right)^2}} = -\arcsin \frac{1}{|x|} + C.
 \end{aligned}$$

$$1684. \int \frac{dx}{(x^2+1)^{\frac{3}{2}}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{dx}{(x^2+1)^{\frac{3}{2}}} &= \int \frac{\operatorname{sgn} x dx}{x^3 \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}}} \\
 &= -\frac{1}{2} \int \left(1 + \frac{1}{x^2}\right)^{-\frac{3}{2}} \operatorname{sgn} x d\left(1 + \frac{1}{x^2}\right) \\
 &= \left(1 + \frac{1}{x^2}\right)^{-\frac{1}{2}} \operatorname{sgn} x + C = \frac{x}{\sqrt{x^2+1}} + C.
 \end{aligned}$$

$$1685. \int \frac{xdx}{(x^2-1)^{\frac{3}{2}}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{xdx}{(x^2-1)^{\frac{3}{2}}} &= \frac{1}{2} \int (x^2-1)^{-\frac{3}{2}} d(x^2-1) \\
 &= -\frac{1}{\sqrt{x^2-1}} + C.
 \end{aligned}$$

$$1686. \int \frac{x^2 dx}{(8x^3+27)^{\frac{2}{3}}}.$$

$$\text{解} \quad \int \frac{x^2 dx}{(8x^3+27)^{\frac{2}{3}}} = \frac{1}{24} \int (8x^3+27)^{-\frac{2}{3}} d(8x^3+27).$$

$$= \frac{1}{8} \sqrt[3]{8x^3 + 27} + C.$$

1687. $\int \frac{dx}{\sqrt{x(1+x)}}.$

解 由 $x(1+x) > 0$ 知: $x > 0$ 或 $x < -1$.

当 $x > 0$ 时,

$$\begin{aligned} \int \frac{dx}{\sqrt{x(1+x)}} &= 2 \int \frac{d(\sqrt{x})}{\sqrt{1+(\sqrt{x})^2}} \\ &= 2 \ln(\sqrt{x} + \sqrt{1+x}) + C; \end{aligned}$$

当 $x < -1$ 时,

$$\begin{aligned} \int \frac{dx}{\sqrt{x(1+x)}} &= - \int \frac{d[-(1+x)]}{\sqrt{(-x)[-(1+x)]}} \\ &= -2 \int \frac{d(\sqrt{-(1+x)})}{\sqrt{1+(\sqrt{-(1+x)})^2}} \\ &= -2 \ln(\sqrt{-x} + \sqrt{-(1+x)}) + C. \end{aligned}$$

总之, 得

$$\int \frac{dx}{\sqrt{x(1+x)}} = 2 \operatorname{sgn} x \cdot \ln(\sqrt{|x|} + \sqrt{|1+x|}) + C.$$

1688. $\int \frac{dx}{\sqrt{x(1-x)}}.$

解 由 $x(1-x) > 0$ 知: $0 < x < 1$. 于是, 得

$$\int \frac{dx}{\sqrt{x(1-x)}} = 2 \int \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}}$$

$$= 2 \arcsin \sqrt{x} + C.$$

$$1689. \int x e^{-x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int x e^{-x^2} dx &= -\frac{1}{2} \int e^{-x^2} d(-x^2) \\ &= -\frac{1}{2} e^{-x^2} + C. \end{aligned}$$

$$1690. \int \frac{e^x dx}{2+e^x}.$$

$$\text{解} \quad \int \frac{e^x dx}{2+e^x} = \int \frac{d(2+e^x)}{2+e^x} = \ln(2+e^x) + C.$$

$$1691. \int \frac{dx}{e^x + e^{-x}}.$$

$$\text{解} \quad \int \frac{dx}{e^x + e^{-x}} = \int \frac{d(e^x)}{1+(e^x)^2} = \arctg(e^x) + C.$$

$$1692. \int \frac{dx}{\sqrt{1+e^{2x}}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{1+e^{2x}}} &= -\int \frac{d(e^{-x})}{\sqrt{1+(e^{-x})^2}} \\ &= -\ln(e^{-x} + \sqrt{1+e^{-2x}}) + C. \end{aligned}$$

$$1693. \int \frac{\ln^2 x}{x} dx.$$

$$\text{解} \quad \int \frac{\ln^2 x}{x} dx = \int \ln^2 x d(\ln x) = \frac{1}{3} \ln^3 x + C.$$

$$1694. \int \frac{dx}{x \ln x \ln(\ln x)}$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x \ln x \ln(\ln x)} &= \int \frac{d(\ln x)}{\ln x \ln(\ln x)} \\ &= \int \frac{d[\ln(\ln x)]}{\ln(\ln x)} = \ln |\ln(\ln x)| + C. \end{aligned}$$

$$1695. \int \sin^5 x \cos x dx.$$

$$\text{解} \quad \int \sin^5 x \cos x dx = \int \sin^4 x d(\sin x) = \frac{1}{6} \sin^6 x + C.$$

$$1696. \int \frac{\sin x}{\sqrt{\cos^3 x}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin x}{\sqrt{\cos^3 x}} dx &= - \int (\cos x)^{-\frac{3}{2}} d(\cos x) \\ &= \frac{2}{\sqrt{\cos x}} + C. \end{aligned}$$

$$1697. \int \operatorname{tg} x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{tg} x dx &= \int \frac{\sin x}{\cos x} dx = - \int \frac{d(\cos x)}{\cos x} \\ &= -\ln |\cos x| + C. \end{aligned}$$

$$1698. \int \operatorname{ctg} x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{ctg} x dx &= \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} \\ &= \ln |\sin x| + C. \end{aligned}$$

$$1699. \int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx.$$

$$\text{解} \int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx$$

$$= \int (\sin x - \cos x)^{-\frac{1}{3}} d(\sin x - \cos x)$$

$$= \frac{3}{2} \sqrt[3]{(\sin x - \cos x)^2} + C = \frac{3}{2} \sqrt[3]{1 - \sin 2x} + C.$$

$$1700. \int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx.$$

解 当 $|a| = |b| \neq 0$ 时,

$$\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$$

$$= \frac{1}{|a|} \int \sin x \cos x dx = \frac{1}{2|a|} \sin^2 x + C_1$$

当 $|a| \neq |b|$ 时,

$$\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx$$

$$= \frac{1}{2} \int \frac{d(\sin^2 x)}{\sqrt{(a^2 - b^2) \sin^2 x + b^2}}$$

$$= \frac{1}{a^2 - b^2} \sqrt{(a^2 - b^2) \sin^2 x + b^2} + C$$

$$= \frac{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}}{a^2 - b^2} + C.$$

$$1701. \int \frac{dx}{\sin^2 x \sqrt[4]{\operatorname{ctg} x}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin^2 x \sqrt[4]{\operatorname{ctg} x}} &= - \int (\operatorname{ctg} x)^{-\frac{1}{4}} d(\operatorname{ctg} x) \\ &= - \frac{4}{3} \sqrt[4]{\operatorname{ctg}^3 x} + C. \end{aligned}$$

$$1702. \int \frac{dx}{\sin^2 x + 2 \cos^2 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin^2 x + 2 \cos^2 x} &= \int \frac{1}{\frac{\cos^2 x}{\operatorname{tg}^2 x + 2}} dx \\ &= \int \frac{d(\operatorname{tg} x)}{\operatorname{tg}^2 x + 2} = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right) + C. \end{aligned}$$

$$1703. \int \frac{dx}{\sin x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin x} &= \int \frac{1}{\frac{2 \cos^2 \frac{x}{2}}{\operatorname{tg} \frac{x}{2}}} dx \\ &= \int \frac{d\left(\operatorname{tg} \frac{x}{2}\right)}{\operatorname{tg} \frac{x}{2}} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C. \end{aligned}$$

$$1704. \int \frac{dx}{\cos x}.$$

$$\text{解} \quad \int \frac{dx}{\cos x} = \int \frac{d\left(x + \frac{\pi}{2}\right)}{\sin\left(x + \frac{\pi}{2}\right)} = \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C.$$

$$1705. \quad \int \frac{dx}{\operatorname{sh} x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\operatorname{sh} x} &= \int \frac{1}{\frac{2\operatorname{ch}^2 \frac{x}{2}}{\operatorname{th} \frac{x}{2}}} dx = \int \frac{d\left(\operatorname{th} \frac{x}{2}\right)}{\operatorname{th} \frac{x}{2}} \\ &= \ln \left| \operatorname{th} \frac{x}{2} \right| + C. \end{aligned}$$

$$1706. \quad \int \frac{dx}{\operatorname{ch} x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\operatorname{ch} x} &= \int \frac{2dx}{e^x + e^{-x}} = 2 \int \frac{d(e^x)}{1 + (e^x)^2} \\ &= 2 \operatorname{arc} \operatorname{tg}(e^x) + C. \end{aligned}$$

$$1707. \quad \int \frac{\operatorname{sh} x \operatorname{ch} x}{\sqrt{\operatorname{sh}^4 x + \operatorname{ch}^4 x}} dx.$$

解 因为

$$\begin{aligned} \operatorname{sh}^4 x + \operatorname{ch}^4 x &= (\operatorname{sh}^2 x + \operatorname{ch}^2 x)^2 - 2\operatorname{sh}^2 x \operatorname{ch}^2 x \\ &= \operatorname{ch}^2 2x - \frac{1}{2} \operatorname{sh}^2 2x = \frac{1 + \operatorname{ch}^2 2x}{2}, \end{aligned}$$

所以, 得

$$\int \frac{\operatorname{sh} x \operatorname{ch} x}{\sqrt{\operatorname{sh}^4 x + \operatorname{ch}^4 x}} dx = \int \frac{\frac{1}{4} d(\operatorname{ch} 2x)}{\frac{1}{\sqrt{2}} \sqrt{1 + \operatorname{ch}^2 2x}}$$

$$= \frac{1}{2\sqrt{2}} \ln(\operatorname{ch} 2x + \sqrt{1 + \operatorname{ch}^2 2x}) + C_1$$

$$= \frac{1}{2\sqrt{2}} \ln\left(\frac{\operatorname{ch} 2x}{\sqrt{2}} + \sqrt{\operatorname{sh}^4 x + \operatorname{ch}^4 x}\right) + C.$$

1708. $\int \frac{dx}{\operatorname{ch}^2 x \cdot \sqrt[3]{\operatorname{th}^2 x}}.$

解 $\int \frac{dx}{\operatorname{ch}^2 x \cdot \sqrt[3]{\operatorname{th}^2 x}} = \int (\operatorname{th} x)^{-\frac{2}{3}} d(\operatorname{th} x)$

$$= 3\sqrt[3]{\operatorname{th} x} + C.$$

1709. $\int \frac{\operatorname{arc} \operatorname{tg} x}{1+x^2} dx.$

解 $\int \frac{\operatorname{arc} \operatorname{tg} x}{1+x^2} dx = \int \operatorname{arc} \operatorname{tg} x d(\operatorname{arc} \operatorname{tg} x)$

$$= \frac{1}{2} (\operatorname{arc} \operatorname{tg} x)^2 + C.$$

1710. $\int \frac{dx}{(\operatorname{arc} \sin x)^2 \sqrt{1-x^2}}.$

解 $\int \frac{dx}{(\operatorname{arc} \sin x)^2 \sqrt{1-x^2}} = \int \frac{d(\operatorname{arc} \sin x)}{(\operatorname{arc} \sin x)^2}$

$$= -\frac{1}{\operatorname{arc} \sin x} + C.$$

$$1711. \int \sqrt{\frac{\ln(x + \sqrt{1+x^2})}{1+x^2}} dx.$$

$$\begin{aligned} \text{解} \quad & \int \sqrt{\frac{\ln(x + \sqrt{1+x^2})}{1+x^2}} dx \\ &= \int [\ln(x + \sqrt{1+x^2})]^{\frac{1}{2}} d[\ln(x + \sqrt{1+x^2})] \\ &= \frac{2}{3} \ln^{\frac{3}{2}}(x + \sqrt{1+x^2}) + C. \end{aligned}$$

$$1712. \int \frac{x^2+1}{x^4+1} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{x^2+1}{x^4+1} dx = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{d(x - \frac{1}{x})}{(x - \frac{1}{x})^2 + 2} \\ &= -\frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{x^2-1}{x\sqrt{2}} + C. \end{aligned}$$

$$1713. \int \frac{x^2-1}{x^4+1} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{x^2-1}{x^4+1} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{d(x + \frac{1}{x})}{(x + \frac{1}{x})^2 - 2} \\ &= \frac{1}{2\sqrt{2}} \ln \left(\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1} \right) + C. \end{aligned}$$

$$1714^+. \int \frac{x^{14} dx}{(x^6+1)^4}.$$

$$\begin{aligned}
\text{解} \quad \int \frac{x^{14} dx}{(x^5+1)^4} &= \int \frac{x^{14} dx}{x^{20}(1+x^{-5})^4} \\
&= -\frac{1}{5} \int (1+x^{-5})^{-4} d(1+x^{-5}) \\
&= \frac{1}{15} (1+x^{-5})^{-3} + C_1 = \frac{x^{15}}{15(x^5+1)^3} + C_1 \\
&= \frac{(x^5+1)^3 - 3x^{10} - 3x^5 - 1}{15(x^5+1)^3} + C_1 \\
&= -\frac{3x^{10} + 3x^5 + 1}{15(x^5+1)^3} + C.
\end{aligned}$$

$$1715. \int \frac{x^{\frac{n}{2}} dx}{\sqrt{1+x^{n+2}}}.$$

解 当 $n = -2$ 时,

$$\int \frac{x^{\frac{n}{2}}}{\sqrt{1+x^{n+2}}} dx = \int \frac{dx}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \ln|x| + C,$$

当 $n \neq -2$ 时,

$$\begin{aligned}
\int \frac{x^{\frac{n}{2}}}{\sqrt{1+x^{n+2}}} dx &= \frac{2}{n+2} \int \frac{d(x^{\frac{n+2}{2}})}{\sqrt{1+(x^{\frac{n+2}{2}})^2}} \\
&= \frac{2}{n+2} \ln \left(x^{\frac{n+2}{2}} + \sqrt{1+x^{n+2}} \right) + C.
\end{aligned}$$

$$1716^+. \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx &= \frac{1}{2} \int \ln \frac{1+x}{1-x} d\left(\ln \frac{1+x}{1-x}\right) \\ &= \frac{1}{4} \ln^2 \frac{1+x}{1-x} + C. \end{aligned}$$

$$1717. \int \frac{\cos x dx}{\sqrt{2+\cos 2x}}$$

$$\begin{aligned} \text{解} \quad \int \frac{\cos x dx}{\sqrt{2+\cos 2x}} &= \int \frac{d(\sin x)}{\sqrt{3-2\sin^2 x}} \\ &= \frac{1}{\sqrt{2}} \operatorname{arc} \sin\left(\sqrt{\frac{2}{3}} \sin x\right) + C. \end{aligned}$$

$$1718. \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx &= \frac{1}{2} \int \frac{\sin 2x dx}{1 - \frac{1}{2} \sin^2 2x} \\ &= -\frac{1}{4} \int \frac{d(\cos 2x)}{1 + \cos^2 2x} = -\frac{1}{2} \operatorname{arc} \operatorname{tg}(\cos 2x) + C. \end{aligned}$$

$$1719. \int \frac{2^x \cdot 3^x}{9^x - 4^x} dx.$$

$$\text{解} \quad \int \frac{2^x \cdot 3^x}{9^x - 4^x} dx = \int \frac{\left(\frac{3}{2}\right)^x}{\left[\left(\frac{3}{2}\right)^x\right]^2 - 1} dx$$

$$\begin{aligned}
&= \frac{1}{\ln 3 - \ln 2} \int \frac{d\left[\left(\frac{3}{2}\right)^x\right]}{\left[\left(\frac{3}{2}\right)^x\right]^2 - 1} \\
&= \frac{1}{2(\ln 3 - \ln 2)} \ln \left| \frac{3^x - 2^x}{3^x + 2^x} \right| + C.
\end{aligned}$$

1720. $\int \frac{x dx}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}}.$

解
$$\begin{aligned}
&\int \frac{x dx}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} \\
&= \frac{1}{2} \int \frac{d(1+x^2)}{\sqrt{1+x^2} \cdot \sqrt{1+x^2} + \sqrt{1+x^2}^3} = \int \frac{d(1+\sqrt{1+x^2})}{\sqrt{1+\sqrt{1+x^2}}} \\
&= 2\sqrt{1+\sqrt{1+x^2}} + C.
\end{aligned}$$

用分项积分法计算下列积分:

1721. $\int x^2(2-3x^2)^2 dx.$

解
$$\begin{aligned}
&\int x^2(2-3x^2)^2 dx = \int (4x^2 - 12x^4 + 9x^6) dx \\
&= \frac{4}{3}x^3 - \frac{12}{5}x^5 + \frac{9}{7}x^7 + C.
\end{aligned}$$

1722. $\int \frac{1+x}{1-x} dx.$

解
$$\begin{aligned}
&\int \frac{1+x}{1-x} dx = \int \left(-1 + \frac{2}{1-x}\right) dx \\
&= -x - 2\ln|1-x| + C.
\end{aligned}$$

$$1723. \int \frac{x^2}{1+x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2}{1+x} dx &= \int \left(x-1 + \frac{1}{1+x} \right) dx \\ &= \frac{1}{2}x^2 - x + \ln|1+x| + C. \end{aligned}$$

$$1724. \int \frac{x^3}{3+x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3}{3+x} dx &= \int \left(x^2 - 3x + 9 - \frac{27}{3+x} \right) dx \\ &= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln|3+x| + C. \end{aligned}$$

$$1725. \int \frac{(1+x)^2}{1+x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{(1+x)^2}{1+x^2} dx &= \int \left(1 + \frac{2x}{1+x^2} \right) dx \\ &= x + \ln(1+x^2) + C. \end{aligned}$$

$$1726. \int \frac{(2-x)^2}{2-x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{(2-x)^2}{2-x^2} dx &= \int \frac{(x^2-2)-4x+6}{2-x^2} dx \\ &= \int \left(-1 - \frac{4x}{2-x^2} + \frac{6}{2-x^2} \right) dx \\ &= -x + 2\ln|2-x^2| + \frac{3}{\sqrt{2}} \ln \left| \frac{\sqrt{2}+x}{\sqrt{2}-x} \right| + C. \end{aligned}$$

$$1727. \int \frac{x^2}{(1-x)^{100}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2}{(1-x)^{100}} dx &= \int \frac{(x-1+1)^2}{(1-x)^{100}} dx \\ &= \int \left[(1-x)^{-98} - 2(1-x)^{-99} + (1-x)^{-100} \right] dx \\ &= \frac{1}{97(1-x)^{97}} - \frac{1}{49(1-x)^{98}} + \frac{1}{99(1-x)^{99}} + C. \end{aligned}$$

$$1728. \int \frac{x^5}{x+1} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^5}{x+1} dx &= \int \left(x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1} \right) dx \\ &= \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|1+x| + C. \end{aligned}$$

$$1729. \int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}} \\ &= \int \frac{1}{2}(\sqrt{x+1} - \sqrt{x-1}) dx \\ &= \frac{1}{3} \left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}} \right] + C. \end{aligned}$$

$$1730. \int x\sqrt{2-5x} dx.$$

$$\text{解} \quad \int x\sqrt{2-5x} dx$$

$$\begin{aligned}
&= \int \left[-\frac{1}{5}(2-5x) + \frac{2}{5} \right] (2-5x)^{\frac{1}{2}} dx \\
&= \int \left[-\frac{1}{5}(2-5x)^{\frac{3}{2}} + \frac{2}{5}(2-5x)^{\frac{1}{2}} \right] dx \\
&= \frac{2}{125}(2-5x)^{\frac{5}{2}} - \frac{4}{75}(2-5x)^{\frac{3}{2}} + C \\
&= -\frac{8+30x}{375}(2-5x)^{\frac{3}{2}} + C.
\end{aligned}$$

1731. $\int \frac{x dx}{\sqrt[3]{1-3x}}$.

解 $\int \frac{x dx}{\sqrt[3]{1-3x}} = -\frac{1}{3} \int \frac{(1-3x)^{-1}}{(1-3x)^{\frac{1}{3}}} dx$

$$\begin{aligned}
&= -\frac{1}{3} \int \left[(1-3x)^{\frac{2}{3}} - (1-3x)^{-\frac{1}{3}} \right] dx \\
&= \frac{1}{15}(1-3x)^{\frac{5}{3}} - \frac{1}{6}(1-3x)^{\frac{2}{3}} + C \\
&= -\frac{1+2x}{10}(1-3x)^{\frac{2}{3}} + C.
\end{aligned}$$

1732. $\int x^3 \sqrt[3]{1+x^2} dx$.

解 $\int x^3 \sqrt[3]{1+x^2} dx$

$$= \frac{1}{2} \int [(x^2+1)-1](1+x^2)^{\frac{1}{3}} d(1+x^2)$$

$$= \frac{1}{2} \int \left[(1+x^2)^{\frac{4}{3}} - (1+x^2)^{\frac{1}{3}} \right] d(1+x^2)$$

$$= \frac{3}{14} (1+x^2)^{\frac{7}{3}} - \frac{3}{8} (1+x^2)^{\frac{4}{3}} + C$$

$$= \frac{12x^2-9}{56} (1+x^2)^{\frac{4}{3}} + C.$$

1733. $\int \frac{dx}{(x-1)(x+3)}.$

解 $\int \frac{dx}{(x-1)(x+3)} = \frac{1}{4} \int \left(\frac{1}{x-1} - \frac{1}{x+3} \right) dx$

$$= \frac{1}{4} \ln \left| \frac{x-1}{x+3} \right| + C.$$

1734. $\int \frac{dx}{x^2+x-2}.$

解 $\int \frac{dx}{x^2+x-2} = \frac{1}{3} \int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx$

$$= \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C.$$

1735. $\int \frac{dx}{(x^2+1)(x^2+2)}.$

解 $\int \frac{dx}{(x^2+1)(x^2+2)} = \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2} \right) dx$

$$= \operatorname{arc} \operatorname{tg} x - \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{x}{\sqrt{2}} + C.$$

$$1736. \int \frac{dx}{(x^2-2)(x^2+3)}.$$

$$\begin{aligned} \text{解} \int \frac{dx}{(x^2-2)(x^2+3)} &= \frac{1}{5} \int \left(\frac{1}{x^2-2} - \frac{1}{x^2+3} \right) dx \\ &= \frac{1}{10\sqrt{2}} \ln \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| - \frac{1}{5\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{x}{\sqrt{3}} + C. \end{aligned}$$

$$1737. \int \frac{x dx}{(x+2)(x+3)}.$$

$$\begin{aligned} \text{解} \int \frac{x dx}{(x+2)(x+3)} &= \int \left(\frac{3}{x+3} - \frac{2}{x+2} \right) dx \\ &= \ln \frac{|x+3|^3}{(x+2)^2} + C. \end{aligned}$$

$$1738. \int \frac{x dx}{x^4 + 3x^2 + 2}.$$

$$\begin{aligned} \text{解} \int \frac{x dx}{x^4 + 3x^2 + 2} &= \frac{1}{2} \int \frac{d(x^2)}{(x^2+1)(x^2+2)} \\ &= \frac{1}{2} \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2} \right) d(x^2) \\ &= \frac{1}{2} \ln \frac{x^2+1}{x^2+2} + C. \end{aligned}$$

$$1739. \int \frac{dx}{(x+a)^2(x+b)^2} \quad (a \neq b).$$

$$\text{解} \int \frac{dx}{(x+a)^2(x+b)^2}$$

$$\begin{aligned}
&= \frac{1}{(a-b)^2} \int \left(\frac{1}{x+a} - \frac{1}{x+b} \right)^2 dx \\
&= \frac{1}{(a-b)^2} \int \left[\frac{1}{(x+a)^2} + \frac{1}{(x+b)^2} - \frac{2}{(x+a)(x+b)} \right] dx \\
&= -\frac{1}{(a-b)^2} \left(\frac{1}{x+a} + \frac{1}{x+b} \right) - \frac{2}{(a-b)^2} \int \frac{dx}{(x+a)(x+b)} \\
&= -\frac{2x+a+b}{(a-b)^2(x+a)(x+b)} + \frac{2}{(a-b)^3} \ln \left| \frac{x+a}{x+b} \right| + C.
\end{aligned}$$

1740. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$ ($|a| \neq |b|$).

解 $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$$\begin{aligned}
&= \frac{1}{a^2-b^2} \int \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} \right) dx \\
&= \frac{1}{a^2-b^2} \left(\frac{1}{b} \operatorname{arc} \operatorname{tg} \frac{x}{b} - \frac{1}{a} \operatorname{arc} \operatorname{tg} \frac{x}{a} \right) + C.
\end{aligned}$$

1741. $\int \sin^2 x dx$.

解 $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx$

$$= \frac{x}{2} - \frac{1}{4} \sin 2x + C.$$

1742. $\int \cos^2 x dx$.

解 $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx$

$$= \frac{x}{2} + \frac{1}{4} \sin 2x + C.$$

1743. $\int \sin x \cdot \sin(x+a) dx.$

解 $\int \sin x \cdot \sin(x+a) dx$
 $= \frac{1}{2} \int [\cos a - \cos(2x+a)] dx$
 $= \frac{x}{2} \cos a - \frac{1}{4} \sin(2x+a) + C.$

1744. $\int \sin 3x \cdot \sin 5x dx.$

解 $\int \sin 3x \cdot \sin 5x dx = \frac{1}{2} \int (\cos 2x - \cos 8x) dx$
 $= \frac{1}{4} \sin 2x - \frac{1}{16} \sin 8x + C.$

1745. $\int \cos \frac{x}{2} \cdot \cos \frac{x}{3} dx.$

解 $\int \cos \frac{x}{2} \cdot \cos \frac{x}{3} dx = \frac{1}{2} \int (\cos \frac{5x}{6} + \cos \frac{x}{6}) dx$
 $= \frac{3}{5} \sin \frac{5x}{6} + 3 \sin \frac{x}{6} + C.$

1746. $\int \sin(2x - \frac{\pi}{6}) \cdot \cos(3x + \frac{\pi}{4}) dx.$

解 $\int \sin(2x - \frac{\pi}{6}) \cdot \cos(3x + \frac{\pi}{4}) dx$

$$\begin{aligned}
&= \frac{1}{2} \int \left[\sin\left(5x + \frac{\pi}{12}\right) - \sin\left(x + \frac{5\pi}{12}\right) \right] dx \\
&= -\frac{1}{10} \cos\left(5x + \frac{\pi}{12}\right) + \frac{1}{2} \cos\left(x + \frac{5\pi}{12}\right) + C.
\end{aligned}$$

1747. $\int \sin^3 x dx.$

$$\begin{aligned}
\text{解} \quad \int \sin^3 x dx &= \int (\cos^2 x - 1) d(\cos x) \\
&= \frac{1}{3} \cos^3 x - \cos x + C.
\end{aligned}$$

1748. $\int \cos^3 x dx.$

$$\begin{aligned}
\text{解} \quad \int \cos^3 x dx &= \int (1 - \sin^2 x) d(\sin x) \\
&= \sin x - \frac{1}{3} \sin^3 x + C.
\end{aligned}$$

1749. $\int \sin^4 x dx.$

$$\begin{aligned}
\text{解} \quad \int \sin^4 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx \\
&= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\
&= \frac{1}{8} \int (3 - 4\cos 2x + \cos 4x) dx \\
&= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C.
\end{aligned}$$

$$1750. \int \cos^4 x dx.$$

$$\begin{aligned} \text{解} \quad \int \cos^4 x dx &= \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\ &= \frac{1}{8} \int (3 + 4\cos 2x + \cos 4x) dx \\ &= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \end{aligned}$$

$$1751. \int \operatorname{ctg}^2 x dx.$$

$$\text{解} \quad \int \operatorname{ctg}^2 x dx = \int (\operatorname{csc}^2 x - 1) dx = -\operatorname{ctg} x - x + C.$$

$$1752. \int \operatorname{tg}^3 x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{tg}^3 x dx &= \int \operatorname{tg} x \cdot (\sec^2 x - 1) dx \\ &= \int \operatorname{tg} x d(\operatorname{tg} x) - \int \operatorname{tg} x dx \\ &= \frac{1}{2}\operatorname{tg}^2 x + \ln |\cos x| + C, \end{aligned}$$

其中第二个积分见1697题。

$$1753. \int \sin^2 3x \cdot \sin^3 2x dx.$$

解 因为

$$\sin^2 3x \cdot \sin^3 2x = \frac{1}{2}(1 - \cos 6x) \cdot \frac{1}{4}(3\sin 2x - \sin 6x)$$

$$= \frac{1}{8} (3\sin 2x - 3\cos 6x \cdot \sin 2x - \sin 6x + \sin 6x \cdot \cos 6x)$$

$$= \frac{3}{8} \sin 2x + \frac{3}{16} \sin 4x - \frac{1}{8} \sin 6x - \frac{3}{16} \sin 8x + \frac{1}{16} \sin 12x$$

所以，得

$$\int \sin^2 3x \cdot \sin^3 2x dx = -\frac{3}{16} \cos 2x - \frac{3}{64} \cos 4x \\ + \frac{1}{48} \cos 6x + \frac{3}{128} \cos 8x - \frac{1}{192} \cos 12x + C.$$

$$1754. \int \frac{dx}{\sin^2 x \cdot \cos^2 x}.$$

$$\text{解} \quad \int \frac{dx}{\sin^2 x \cdot \cos^2 x} = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right) dx \\ = -\operatorname{ctg} x + \operatorname{tg} x + C.$$

$$1755. \int \frac{dx}{\sin^2 x \cdot \cos x}.$$

$$\text{解} \quad \int \frac{dx}{\sin^2 x \cdot \cos x} = \int \left(\frac{1}{\cos x} + \frac{\cos x}{\sin^2 x} \right) dx \\ = \int \frac{dx}{\cos x} + \int \frac{d(\sin x)}{\sin^2 x} \\ = \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| - \frac{1}{\sin x} + C,$$

其中第一个积分见1704题。

$$1756. \int \frac{dx}{\sin x \cdot \cos^3 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin x \cdot \cos^3 x} &= \int \left(\frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x} \right) dx \\ &= - \int \frac{d(\cos x)}{\cos^3 x} + \int \frac{d(2x)}{\sin 2x} \\ &= \frac{1}{2\cos^2 x} + \ln |\operatorname{tg} x| + C, \end{aligned}$$

其中第二个积分见1703题.

$$1757. \int \frac{\cos^3 x}{\sin x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\cos^3 x}{\sin x} dx &= \int \frac{1 - \sin^2 x}{\sin x} \cos x dx \\ &= \int \left(\frac{1}{\sin x} - \sin x \right) d(\sin x) \\ &= \ln |\sin x| - \frac{1}{2} \sin^2 x + C. \end{aligned}$$

$$1758. \int \frac{dx}{\cos^4 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\cos^4 x} &= \int \sec^2 x \cdot \frac{dx}{\cos^2 x} = \int (1 + \operatorname{tg}^2 x) d(\operatorname{tg} x) \\ &= \operatorname{tg} x + \frac{1}{3} \operatorname{tg}^3 x + C, \end{aligned}$$

$$1759. \int \frac{dx}{1 + e^x}.$$

$$\text{解 } \int \frac{dx}{1+e^x} = \int \left(1 - \frac{e^x}{1+e^x}\right) dx = x - \ln(1+e^x) + C.$$

$$1760. \int \frac{(1+e^x)^2}{1+e^{2x}} dx.$$

$$\begin{aligned} \text{解 } \int \frac{(1+e^x)^2}{1+e^{2x}} dx &= \int \left(1 + \frac{2e^x}{1+e^{2x}}\right) dx \\ &= x + 2 \operatorname{arc} \operatorname{tg}(e^x) + C. \end{aligned}$$

$$1761. \int \operatorname{sh}^2 x dx.$$

$$\text{解 } \int \operatorname{sh}^2 x dx = \int \frac{\operatorname{ch} 2x - 1}{2} dx = \frac{1}{4} \operatorname{sh} 2x - \frac{x}{2} + C.$$

$$1762. \int \operatorname{ch}^2 x dx.$$

$$\text{解 } \int \operatorname{ch}^2 x dx = \int \frac{\operatorname{ch} 2x + 1}{2} dx = \frac{1}{4} \operatorname{sh} 2x + \frac{x}{2} + C.$$

$$1763. \int \operatorname{sh} x \cdot \operatorname{sh} 2x dx.$$

$$\begin{aligned} \text{解 } \int \operatorname{sh} x \cdot \operatorname{sh} 2x dx &= 2 \int \operatorname{sh}^2 x \operatorname{ch} x dx = 2 \int \operatorname{sh}^2 x d(\operatorname{sh} x) \\ &= \frac{2}{3} \operatorname{sh}^3 x + C. \end{aligned}$$

$$1764. \int \operatorname{ch} x \cdot \operatorname{ch} 3x dx.$$

$$\text{解 } \int \operatorname{ch} x \cdot \operatorname{ch} 3x dx = \frac{1}{2} \int (\operatorname{ch} 4x + \operatorname{ch} 2x) dx$$

$$= \frac{1}{8} \operatorname{sh} 4x + \frac{1}{4} \operatorname{sh} 2x + C.$$

$$1765. \int \frac{dx}{\operatorname{sh}^2 x \cdot \operatorname{ch}^2 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\operatorname{sh}^2 x \cdot \operatorname{ch}^2 x} &= \int \left(\frac{1}{\operatorname{sh}^2 x} - \frac{1}{\operatorname{ch}^2 x} \right) dx \\ &= -(\operatorname{cth} x + \operatorname{th} x) + C. \end{aligned}$$

用适当的代换, 求下列积分:

$$1766. \int x^2 \sqrt[3]{1-x} dx.$$

解 设 $1-x=t$, 则 $x=1-t$, $dx=-dt$, 故得

$$\begin{aligned} \int x^2 \sqrt[3]{1-x} dx &= - \int (1-t)^2 t^{\frac{1}{3}} dt \\ &= - \int \left(t^{\frac{1}{3}} - 2t^{\frac{4}{3}} + t^{\frac{7}{3}} \right) dt \\ &= - \frac{3}{4} t^{\frac{4}{3}} + \frac{6}{7} t^{\frac{7}{3}} - \frac{3}{10} t^{\frac{10}{3}} + C \\ &= - \frac{3}{140} (9 + 12x + 14x^2) (1-x)^{\frac{4}{3}} + C. \end{aligned}$$

$$1767. \int x^3 (1-5x^2)^{10} dx.$$

解 设 $1-5x^2=t$, 则 $x^2 = \frac{1}{5}(1-t)$, 从而 $x^3 dx$

$$= \frac{1}{2} x^2 d(x^2) = \frac{1}{10} (1-t) \left(-\frac{1}{5} \right) dt$$

$$= -\frac{1}{50}(1-t)dt, \text{ 故得}$$

$$\begin{aligned} \int x^3(1-5x^2)^{10}dx &= -\frac{1}{50} \int (t^{10}-t^{11})dt \\ &= -\frac{1}{550}t^{11} + \frac{1}{600}t^{12} + C \\ &= -\frac{1+55x^2}{6600}(1-5x^2)^{11} + C. \end{aligned}$$

1768. $\int \frac{x^2}{\sqrt{2-x}} dx.$

解 设 $2-x=t$, 则 $x=2-t$, $dx=-dt$, 故得

$$\begin{aligned} \int \frac{x^2}{\sqrt{2-x}} dx &= - \int t^{-\frac{1}{2}}(2-t)^2 dt \\ &= - \int \left(4t^{-\frac{1}{2}} - 4t^{\frac{1}{2}} + t^{\frac{3}{2}}\right) dt \\ &= -8t^{\frac{1}{2}} + \frac{8}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}} + C \\ &= -\frac{2}{15}(32+8x+3x^2)\sqrt{2-x} + C. \end{aligned}$$

1769. $\int \frac{x^5}{\sqrt{1-x^2}} dx.$

解 设 $1-x^2=t$, 则 $x^2=1-t$, 从而 $x^5 dx = \frac{1}{2}(x^2)^2$

$$\cdot d(x^2) = -\frac{1}{2}(1-t)^2 dt, \text{ 故得}$$

$$\begin{aligned}
\int \frac{x^5}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int t^{-\frac{1}{2}}(1-t)^2 dt \\
&= -\frac{1}{2} \int (t^{-\frac{1}{2}} - 2t^{\frac{1}{2}} + t^{\frac{3}{2}}) dt \\
&= -t^{\frac{1}{2}} + \frac{2}{3}t^{\frac{3}{2}} - \frac{1}{5}t^{\frac{5}{2}} + C \\
&= -\frac{1}{15}(8+4x^2+3x^4)\sqrt{1-x^2} + C.
\end{aligned}$$

1770. $\int x^5(2-5x^3)^{\frac{2}{3}} dx.$

解 设 $2-5x^3=t$, 则 $x^3 = \frac{1}{5}(2-t)$, 从而

$$x^5 dx = \frac{1}{3}x^3 d(x^3) = -\frac{1}{75}(2-t)dt,$$

故得

$$\begin{aligned}
\int x^5(2-5x^3)^{\frac{2}{3}} dx &= -\frac{1}{75} \int t^{\frac{2}{3}}(2-t) dt \\
&= -\frac{1}{75} \int (2t^{\frac{2}{3}} - t^{\frac{5}{3}}) dt = -\frac{2}{125}t^{\frac{5}{3}} + \frac{1}{200}t^{\frac{8}{3}} + C \\
&= -\frac{6+25x^3}{1000}(2-5x^3)^{\frac{5}{3}} + C.
\end{aligned}$$

1771+. $\int \cos^5 x \sqrt{\sin x} dx.$

解 设 $\sin x = t$, 则 $\cos^5 x dx = (1-\sin^2 x)^2 d(\sin x)$
 $= (1-t^2)^2 dt,$

故得

$$\begin{aligned}\int \cos^6 x \sqrt{\sin x} dx &= \int (1-t^2)^2 t^{\frac{1}{2}} dt \\ &= \int \left(t^{\frac{1}{2}} - 2t^{\frac{5}{2}} + t^{\frac{9}{2}} \right) dt \\ &= \frac{2}{3} t^{\frac{3}{2}} - \frac{4}{7} t^{\frac{7}{2}} + \frac{2}{11} t^{\frac{11}{2}} + C \\ &= \left(\frac{2}{3} - \frac{4}{7} \sin^2 x + \frac{2}{11} \sin^4 x \right) \sqrt{\sin^3 x} + C.\end{aligned}$$

1772. $\int \frac{\sin x \cos^3 x}{1 + \cos^2 x} dx.$

解 设 $\cos^2 x = t$, 则 $\sin x \cos x dx = -\frac{1}{2} dt$, 故得

$$\begin{aligned}\int \frac{\sin x \cos^3 x}{1 + \cos^2 x} dx &= -\frac{1}{2} \int \frac{t}{1+t} dt \\ &= -\frac{1}{2} \int \left(1 - \frac{1}{1+t} \right) dt = -\frac{1}{2} t + \frac{1}{2} \ln(1+t) + C \\ &= -\frac{1}{2} \cos^2 x + \frac{1}{2} \ln(1 + \cos^2 x) + C.\end{aligned}$$

1773. $\int \frac{\sin^2 x}{\cos^6 x} dx.$

解 设 $\operatorname{tg} x = t$, 则 $\frac{1}{\cos^4 x} dx = (1+t^2) dt$, 故得

$$\int \frac{\sin^2 x}{\cos^6 x} dx = \int (t^4 + t^2) dt = \frac{1}{5} t^5 + \frac{1}{3} t^3 + C$$

$$= \frac{1}{5} \operatorname{tg}^5 x + \frac{1}{3} \operatorname{tg}^3 x + C.$$

$$1774. \int \frac{\ln x dx}{x\sqrt{1+\ln x}}.$$

解 设 $1 + \ln x = t$, 则 $\frac{\ln x dx}{x} = (1 + \ln x - 1)d(1 + \ln x)$

$= (t - 1)dt$, 故得

$$\int \frac{\ln x dx}{x\sqrt{1+\ln x}} = \int t^{-\frac{1}{2}}(t-1)dt$$

$$= \int (t^{\frac{1}{2}} - t^{-\frac{1}{2}})dt = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(\ln x - 2)\sqrt{1+\ln x} + C.$$

$$1775. \int \frac{dx}{e^{\frac{x}{2}} + e^x}.$$

解 设 $e^{\frac{x}{2}} = t$, 则 $e^x = t^2$, $dx = \frac{2dt}{t}$, 故得

$$\int \frac{dx}{e^{\frac{x}{2}} + e^x} = 2 \int \frac{dt}{t^2(1+t)} = 2 \int \left(\frac{1-t}{t^2} + \frac{1}{1+t} \right) dt$$

$$= -\frac{2}{t} - 2\ln t + 2\ln(1+t) + C$$

$$= -2e^{-\frac{x}{2}} - x + 2\ln(1+e^{\frac{x}{2}}) + C.$$

$$1776. \int \frac{dx}{\sqrt{1+e^x}}.$$

解 设 $\sqrt{1+e^x}=t$, 则 $x=\ln(t^2-1)$, $dx=\frac{2t}{t^2-1}dt$,

故得

$$\begin{aligned}\int \frac{dx}{\sqrt{1+e^x}} &= 2 \int \frac{dt}{t^2-1} = \ln\left(\frac{t-1}{t+1}\right) + C \\ &= \ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + C = x - 2\ln(1+\sqrt{1+e^x}) + C.\end{aligned}$$

1777. $\int \frac{\arctg\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x}$.

解 设 $\arctg\sqrt{x}=t$, 则 $dt=\frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}dx$, 故得

$$\begin{aligned}\int \frac{\arctg\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x} &= 2 \int t dt = t^2 + C \\ &= (\arctg\sqrt{x})^2 + C.\end{aligned}$$

运用三角的代换 $x=asint$, $x=atgt$, $x=asin^2t$ 等等, 求下列积分 (参数为正的):

1778. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$.

解 由于被积函数的存在域为 $-1 < x < 1$, 因此可设

$x = sint$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$. 从而

$$(1-x^2)^{\frac{3}{2}} = \cos^3 t, \quad dx = cost dt.$$

代入得

$$\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{dt}{\cos^2 t} = \operatorname{tg} t + C$$

$$= \frac{\sin t}{\sqrt{1-\sin^2 t}} + C = \frac{x}{\sqrt{1-x^2}} + C.$$

1779. $\int \frac{x^2 dx}{\sqrt{x-2}}.$

解 被积函数的存在域为 $x > \sqrt{2}$ 及 $x < -\sqrt{2}$, 分别考虑.

(1) 当 $x > \sqrt{2}$ 时, 可设 $x = \sqrt{2} \operatorname{sect}$, 并限制 $0 <$

$t < \frac{\pi}{2}$. 从而

$$\frac{x^2}{\sqrt{x^2-2}} = \frac{2\sec^2 t}{\sqrt{2}\operatorname{tg} t}, \quad dx = \sqrt{2} \operatorname{sect} \cdot \operatorname{tg} t \, dt.$$

代入得

$$\int \frac{x^2 dx}{\sqrt{x^2-2}} = 2 \int \sec^3 t \, dt = 2 \int \frac{d(\sin t)}{(1-\sin^2 t)^2}$$

$$= \frac{1}{2} \int \left(\frac{1}{1+\sin t} + \frac{1}{1-\sin t} \right)^2 d(\sin t)$$

$$= \frac{1}{2} \int \frac{d(1+\sin t)}{(1+\sin t)^2} - \frac{1}{2} \int \frac{d(1-\sin t)}{(1-\sin t)^2} + \int \frac{d(\sin t)}{1-\sin^2 t}$$

$$= \frac{1}{2} \left(-\frac{1}{1-\sin t} - \frac{1}{1+\sin t} \right) + \frac{1}{2} \ln \left(\frac{1+\sin t}{1-\sin t} \right) + C_1$$

$$= \operatorname{tg} t \cdot \operatorname{sect} + \ln(\operatorname{sect} + \operatorname{tg} t) + C_1$$

$$= \frac{x}{2} \sqrt{x^2 - 2} + \ln(x + \sqrt{x^2 - 2}) + C.$$

(2) 当 $x < -\sqrt{2}$ 时, 仍设 $x = \sqrt{2} \sec t$, 但限制 $\pi < t < \frac{3\pi}{2}$. 其余步骤与上相同, 注意到, 此时 $\sec t + \operatorname{tg} t < 0$, 因此在对数符号里要加绝对值, 即结果为 $\frac{x}{2} \sqrt{x^2 - 2} + \ln|x + \sqrt{x^2 - 2}| + C$.

总之, 当 $|x| > \sqrt{2}$ 时,

$$\int \frac{x^2 dx}{\sqrt{x^2 - 2}} = \frac{x}{2} \sqrt{x^2 - 2} + \ln|x + \sqrt{x^2 - 2}| + C.$$

1780. $\int \sqrt{a^2 - x^2} dx.$

解 被积函数的存在域为 $-a \leq x \leq a$, 因此设 $x = a \sin t$, 并限制 $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. 从而

$$\sqrt{a^2 - x^2} = a \cos t, \quad dx = a \cos t dt.$$

代入得

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= a^2 \int \cos^2 t dt = a^2 \left(\frac{t}{2} + \frac{1}{4} \sin 2t \right) + C \\ &= \frac{a^2}{2} t + \frac{a^2}{2} \sin t \cos t + C \\ &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C. \end{aligned}$$

*) 利用1742题的结果.

1781. $\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}}.$

解 被积函数的存在域为 $-\infty < x < +\infty$, 因此可设 $x = a \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$. 从而

$$(x^2 + a^2)^{\frac{3}{2}} = a^3 \sec^3 t, \quad dx = a \sec^2 t dt.$$

代入得

$$\begin{aligned} \int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}} &= \frac{1}{a^2} \int \cos t dt = \frac{1}{a^2} \sin t + C \\ &= \frac{1}{a^2} \cdot \frac{\operatorname{tg} t}{\sqrt{1 + \operatorname{tg}^2 t}} + C = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C. \end{aligned}$$

1782. $\int \sqrt{\frac{a+x}{a-x}} dx.$

解 被积函数的存在域为 $-a \leq x < a$, 因此可设 $x = a \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$. 从而

$$\sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{1+\sin t}{1-\sin t}} = \frac{1+\sin t}{\cos t}, \quad dx = a \cos t dt.$$

代入得

$$\begin{aligned} \int \sqrt{\frac{a+x}{a-x}} dx &= a \int (1 + \sin t) dt = a(t - \cos t) + C \\ &= a \operatorname{arc} \sin \frac{x}{a} - \sqrt{a^2 - x^2} + C \quad (-a < x < a). \end{aligned}$$

注意, 上式在端点 $x = -a$ 也成立. 即函数 $F(x)$

$= a \operatorname{arc} \sin \frac{x}{a} - \sqrt{a^2 - x^2}$ 在点 $x = -a$ 的 (右) 导数等于

被积函数 $f(x) = \sqrt{\frac{a+x}{a-x}}$ 在点 $x = -a$ 之值。事实上，

由于 $F(x)$ 和 $f(x)$ 都在 $-a \leq x < a$ 连续，且 $F'(x) = f(x)$ 在 $-a < x < a$ 成立。故由中值定理，知当 $-a < x < a$ 时，有

$$\frac{F(x) - F(-a)}{x + a} = F'(\xi) = f(\xi), \quad -a < \xi < x.$$

由此可知，（右）导数

$$\begin{aligned} F'(-a) &= \lim_{x \rightarrow -a+0} \frac{F(x) - F(-a)}{x + a} \\ &= \lim_{\xi \rightarrow -a+0} f(\xi) = f(-a). \end{aligned}$$

下面有些题目在端点的情况可类似地进行讨论，从略。

1783. $\int x \sqrt{\frac{x}{2a-x}} dx.$

解 被积函数的存在域为 $0 \leq x < 2a$ ，因此可设 x

$= 2a \sin^2 t$ ，并限制 $0 \leq t < \frac{\pi}{2}$ 。从而

$$x \sqrt{\frac{x}{2a-x}} = \frac{2a \sin^3 t}{\cos t}, \quad dx = 4a \sin t \cos t dt.$$

代入得

$$\begin{aligned} \int x \sqrt{\frac{x}{2a-x}} &= 8a^2 \int \sin^4 t dt \\ &= 8a^2 \left(\frac{3}{8} t - \frac{1}{4} \sin 2t + \frac{1}{32} \sin 4t \right) + C. \end{aligned}$$

注意到 $\sin 2t = 2 \sin t \cos t = 2\sqrt{\frac{x}{2a}} \cdot \sqrt{1 - \frac{x}{2a}} = \frac{1}{a}$

$\cdot \sqrt{x(2a-x)}$ 及 $\sin 4t = 2 \sin 2t \cos 2t = 4 \sin t \cos t (1$

$- 2 \sin^2 t) = \frac{2}{a^2} (a-x) \sqrt{x(2a-x)}$, 最后得

$$\int x \sqrt{\frac{x}{2a-x}} dx = 3a^2 \arcsin \sqrt{\frac{x}{2a}}$$

$$- 2a^2 \cdot \frac{1}{a} \sqrt{x(2a-x)} + \frac{1}{4} a^2 \cdot \frac{2}{a^2} (a-x) \sqrt{x(2a-x)} + C$$

$$= 3a^2 \arcsin \sqrt{\frac{x}{2a}} - \frac{3a+x}{2} \sqrt{x(2a-x)} + C.$$

*) 利用1749题的结果。

1784. $\int \frac{dx}{\sqrt{(x-a)(b-x)}}.$

解 不妨设 $a < b$. 被积函数的存在域为 $a < x < b$, 因此可设 $x-a = (b-a) \sin^2 t$, 并限制 $0 < t < \frac{\pi}{2}$. 从而

$$\sqrt{(x-a)(b-x)} = (b-a) \sin t \cos t,$$

$$dx = 2(b-a) \sin t \cos t dt.$$

代入得

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \int dt = 2t + C$$

$$= 2 \arcsin \sqrt{\frac{x-a}{b-a}} + C.$$

$$1785. \int \sqrt{(x-a)(b-x)} dx.$$

解 与上题相同, 作同一代换, 并注意到 $\sin 4t = 4 \sin t$

$$\cdot \cos t (1 - 2 \sin^2 t) = 4 \sqrt{\frac{x-a}{b-a}} \cdot \sqrt{1 - \frac{x-a}{b-a}} \left(1 - 2 \cdot \frac{x-a}{b-a}\right)$$

$$= -4 \cdot \frac{2x - (a+b)}{(b-a)^2} \sqrt{(x-a)(b-x)}, \text{ 即得}$$

$$\int \sqrt{(x-a)(b-x)} dx = 2(b-a)^2 \int \sin^2 t \cos^2 t dt$$

$$= \frac{(b-a)^2}{2} \int \sin^2 2t dt = \frac{(b-a)^2}{4} \int (1 - \cos 4t) dt$$

$$= \frac{(b-a)^2}{4} \left(t - \frac{1}{4} \sin 4t\right) + C$$

$$= \frac{(b-a)^2}{4} \arcsin \sqrt{\frac{x-a}{b-a}}$$

$$+ \frac{2x - (a+b)}{4} \sqrt{(x-a)(b-x)} + C.$$

用双曲线代换 $x = a \operatorname{sh} t$, $x = a \operatorname{ch} t$ 等等, 求下列积分(参数为正的):

$$1786. \int \sqrt{a^2 + x^2} dx.$$

解 被积函数的存在域为 $-\infty < x < +\infty$, 因此可设 $x = a \operatorname{sh} t$. 从而

$$\sqrt{a^2 + x^2} = a \operatorname{ch} t, \quad dx = a \operatorname{ch} t dt.$$

代入得

$$\int \sqrt{a^2+x^2} dx = a^2 \int \operatorname{ch}^2 t dt$$

$$= a^2 \left(\frac{t}{2} + \frac{1}{4} \operatorname{sh} 2t \right) + C_1.$$

注意到 $x + \sqrt{a^2+x^2} = a (\operatorname{sh} t + \operatorname{ch} t) = a e^t$, 即 $t =$

$$\ln \frac{x + \sqrt{a^2+x^2}}{a} \text{ 及 } \operatorname{sh} 2t = 2 \operatorname{sh} t \operatorname{ch} t = \frac{2x\sqrt{a^2+x^2}}{a^2},$$

最后得

$$\int \sqrt{a^2+x^2} dx = \frac{a^2}{2} \ln(x + \sqrt{a^2+x^2})$$

$$+ \frac{x}{2} \sqrt{a^2+x^2} + C.$$

*) 利用1762题的结果.

1787. $\int \frac{x^2}{\sqrt{a^2+x^2}} dx.$

解 与上题相同, 设 $x = a \operatorname{sh} t$, 则

$$\frac{x^2}{\sqrt{a^2+x^2}} = \frac{a \operatorname{sh}^2 t}{\operatorname{ch} t}, \quad dx = a \operatorname{ch} t dt.$$

代入得

$$\int \frac{x^2}{\sqrt{a^2+x^2}} dx = a^2 \int \operatorname{sh}^2 t dt$$

$$= a^2 \left(\frac{1}{4} \operatorname{sh} 2t - \frac{t}{2} \right) + C_1$$

$$= \frac{x}{2} \sqrt{a^2 + x^2} - \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C.$$

*) 利用1761题的结果.

$$1788^+. \int \sqrt{\frac{x-a}{x+a}} dx.$$

解 被积函数的存在域为 $x \geq a$ 及 $x < -a$.

(1) 当 $x > a$ 时, 可设 $x = a \operatorname{ch} t$, 并限制 $t > 0$. 从而

$$\sqrt{\frac{x-a}{x+a}} = \frac{\operatorname{ch} t - 1}{\operatorname{sh} t}, \quad dx = a \operatorname{sh} t dt.$$

代入得

$$\begin{aligned} \int \sqrt{\frac{x-a}{x+a}} dx &= a \int (\operatorname{ch} t - 1) dt \\ &= a \operatorname{sh} t - at + C_1 = a \sqrt{\operatorname{ch}^2 t - 1} - at + C_1 \\ &= a \sqrt{\left(\frac{x}{a}\right)^2 - 1} - a \ln\left(\sqrt{\left(\frac{x}{a}\right)^2 - 1} + \frac{x}{a}\right) + C_1 \\ &= \sqrt{x^2 - a^2} - a \ln(\sqrt{x^2 - a^2} + x) + C_2 \\ &= \sqrt{x^2 - a^2} - 2a \ln(\sqrt{x-a} + \sqrt{x+a}) + C. \end{aligned}$$

(2) 当 $x < -a$ 时, 可设 $x = -a \operatorname{ch} t$, 并限制 $t > 0$. 从而

$$\sqrt{\frac{x-a}{x+a}} = -\frac{\operatorname{ch} t + 1}{\operatorname{sh} t}, \quad dx = -a \operatorname{sh} t dt.$$

代入得

$$\begin{aligned}
\int \sqrt{\frac{x-a}{x+a}} dx &= -a \int (\operatorname{ch}t + 1) dt = -a \operatorname{sh}t - at + C_1 \\
&= -a \sqrt{\left(\frac{x}{a}\right)^2 - 1} - a \ln \left(\sqrt{\left(\frac{x}{a}\right)^2 - 1} - \frac{x}{a} \right) + C_1 \\
&= -\sqrt{x^2 - a^2} - a \ln(\sqrt{x^2 - a^2} - x) + C_2 \\
&= -\sqrt{x^2 - a^2} - 2a \ln(\sqrt{-x+a} + \sqrt{-x-a}) + C.
\end{aligned}$$

总之，当 $|x| > a$ 时，

$$\begin{aligned}
\int \sqrt{\frac{x-a}{x+a}} dx &= \operatorname{sgn}x \cdot \sqrt{x^2 - a^2} \\
&\quad - 2a \ln(\sqrt{|x-a|} + \sqrt{|x+a|}) + C.
\end{aligned}$$

1789. $\int \frac{dx}{\sqrt{(x+a)(x+b)}}$

解 不妨设 $a < b$ 。被积函数的存在域为 $x > -a$ 及 $x < -b$ 。

(1) 当 $x > -a$ 时，可设 $x+a = (b-a) \operatorname{sh}^2 t$ ，并限制 $t > 0$ ，从而

$$\begin{aligned}
\sqrt{(x+a)(x+b)} &= (b-a) \operatorname{sh}t \operatorname{ch}t, \quad dx \\
&= 2(b-a) \operatorname{sh}t \operatorname{ch}t dt.
\end{aligned}$$

代入得

$$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = 2 \int dt = 2t + C_1.$$

注意到 $\sqrt{x+a} + \sqrt{x+b} = \sqrt{b-a}(\operatorname{sh}t + \operatorname{ch}t) = \sqrt{b-a}$

e^t ，就有 $t = \ln \frac{\sqrt{x+a} + \sqrt{x+b}}{\sqrt{b-a}}$ ，最后得

$$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = 2 \ln(\sqrt{x+a} + \sqrt{x+b}) + C.$$

(2) 当 $x < -b$ 时, 可设 $x+b = (a-b)\text{sh}^2 t$, 并限制 $t > 0$. 从而

$$\begin{aligned} \sqrt{(x+a)(x+b)} &= (b-a)\text{sh}t\text{ch}t, \quad dx \\ &= -(b-a)2\text{sh}t\text{ch}t\,dt. \end{aligned}$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt{(x+a)(x+b)}} &= -2 \int dt = -2t + C_1 \\ &= -2 \ln(\sqrt{-(x+a)} + \sqrt{-(x+b)}) + C. \end{aligned}$$

总之,

$$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \begin{cases} 2 \ln(\sqrt{x+a} + \sqrt{x+b}), & \text{若 } x+a > 0 \text{ 及 } x+b > 0; \\ -2 \ln(\sqrt{-x-a} + \sqrt{-x-b}), & \text{若 } x+a < 0 \text{ 及 } x+b < 0. \end{cases}$$

1790. $\int \sqrt{(x+a)(x+b)} dx.$

解 与上题相同, 作同一代换, 只是在求积分的过程中变动个别地方. 今以 $x > -a$ 时为例, 解法如下:

$$\begin{aligned} \int \sqrt{(x+a)(x+b)} dx &= 2(b-a)^2 \int \text{sh}^2 t \text{ch}^2 t dt \\ &= \frac{1}{2}(b-a)^2 \int \text{sh}^2 2t dt \\ &= \frac{1}{4}(b-a)^2 \int (\text{ch}4t - 1) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}(b-a)^2 \left(\frac{1}{4} \operatorname{sh} 4t - t \right) + C_1 \\
&= \frac{1}{4}(b-a)^2 [\operatorname{sh} t \operatorname{ch} t (1 + 2\operatorname{sh}^2 t) - t] + C_1 \\
&= \frac{1}{4}(b-a)^2 \left[\sqrt{\frac{x+a}{b-a}} \cdot \sqrt{1 + \frac{x+a}{b-a}} \left(1 + 2 \cdot \frac{x+a}{b-a} \right) \right. \\
&\quad \left. - \ln(\sqrt{x+a} + \sqrt{x+b}) \right] + C \\
&= \frac{2x+a+b}{4} \sqrt{(x+a)(x+b)} \\
&\quad - \frac{(b-a)^2}{4} \ln(\sqrt{x+a} + \sqrt{x+b}) + C.
\end{aligned}$$

至于当 $x < -b$ 时，与上题类似，只是将结果改成

$$\begin{aligned}
&\frac{2x+a+b}{4} \sqrt{(x+a)(x+b)} + \frac{(b-a)^2}{4} \ln(\sqrt{-x-a} \\
&+ \sqrt{-x-b}) + C, \text{ 此处不再写出解法步骤.}
\end{aligned}$$

总之，

$$\begin{aligned}
&\int \sqrt{(x+a)(x+b)} dx \\
&= \begin{cases} \frac{2x+a+b}{4} \sqrt{(x+a)(x+b)} - \frac{(b-a)^2}{4} \ln(\sqrt{x+a} \\ + \sqrt{x+b}) + C, \text{ 若 } x+a > 0 \text{ 及 } x+b > 0; \\ \frac{2x+a+b}{4} \sqrt{(x+a)(x+b)} + \frac{(b-a)^2}{4} \ln(\sqrt{-x-a} \\ + \sqrt{-x-b}) + C, \text{ 若 } x+a < 0 \text{ 及 } x+b < 0. \end{cases}
\end{aligned}$$

用分部积分法，求下列积分：

1791. $\int \ln x dx.$

解 $\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x(\ln x - 1) + C.$

1792. $\int x^n \ln x dx \quad (n \neq -1).$

解 $\int x^n \ln x dx = \frac{1}{n+1} \int \ln x d(x^{n+1}) = \frac{1}{n+1} x^{n+1} \ln x$
 $- \frac{1}{n+1} \int x^{n+1} \cdot \frac{1}{x} dx = \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C.$

1793. $\int \left(\frac{\ln x}{x} \right)^2 dx.$

解 $\int \left(\frac{\ln x}{x} \right)^2 dx = - \int \ln^2 x d\left(\frac{1}{x}\right)$
 $= - \frac{1}{x} \ln^2 x + \int \frac{1}{x} \cdot 2 \ln x \cdot \frac{1}{x} dx$
 $= - \frac{1}{x} \ln^2 x - 2 \int \ln x d\left(\frac{1}{x}\right) = - \frac{1}{x} \ln^2 x - \frac{2}{x} \ln x$
 $+ 2 \int \frac{1}{x} \cdot \frac{1}{x} dx = - \frac{1}{x} (\ln^2 x + 2 \ln x + 2) + C.$

1794. $\int \sqrt{x} \ln^2 x dx.$

解 $\int \sqrt{x} \ln^2 x dx = \frac{2}{3} \int \ln^2 x d\left(x^{\frac{3}{2}}\right)$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln^2x - \frac{4}{3}\int x^{\frac{3}{2}}\ln x \cdot \frac{1}{x}dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln^2x - \frac{8}{9}\int \ln x d(x^{\frac{3}{2}})$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln^2x - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{8}{9}\int x^{\frac{3}{2}} \cdot \frac{1}{x}dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\left(\ln^2x - \frac{4}{3}\ln x + \frac{8}{9}\right) + C.$$

1795. $\int xe^{-x}dx.$

解 $\int xe^{-x}dx = -\int xd(e^{-x}) = -xe^{-x} + \int e^{-x}dx$
 $= -e^{-x}(x+1) + C.$

1796. $\int x^2e^{-2x}dx.$

解 $\int x^2e^{-2x}dx = -\frac{1}{2}\int x^2d(e^{-2x})$
 $= -\frac{1}{2}x^2e^{-2x} + \frac{1}{2}\int e^{-2x} \cdot 2xdx$
 $= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}\int xd(e^{-2x})$
 $= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx$
 $= -\frac{1}{2}e^{-2x}\left(x^2 + x + \frac{1}{2}\right) + C.$

$$1797. \int x^3 e^{-x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int x^3 e^{-x^2} dx &= -\frac{1}{2} \int x^2 d(e^{-x^2}) \\ &= -\frac{1}{2} x^2 e^{-x^2} + \frac{1}{2} \int e^{-x^2} d(x^2) = -\frac{x^2+1}{2} e^{-x^2} + C. \end{aligned}$$

$$1798. \int x \cos x dx.$$

$$\begin{aligned} \text{解} \quad \int x \cos x dx &= \int x d(\sin x) \\ &= x \sin x - \int \sin x dx = x \sin x + \cos x + C. \end{aligned}$$

$$1799. \int x^2 \sin 2x dx.$$

$$\begin{aligned} \text{解} \quad \int x^2 \sin 2x dx &= -\frac{1}{2} \int x^2 d(\cos 2x) \\ &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} \int x d(\sin 2x) \\ &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{2x^2-1}{4} \cos 2x + \frac{1}{2} x \sin 2x + C. \end{aligned}$$

$$1800. \int x \operatorname{sh} x dx.$$

$$\begin{aligned} \text{解} \quad \int x \operatorname{sh} x dx &= \int x d(\operatorname{ch} x) \\ &= x \operatorname{ch} x - \int \operatorname{ch} x dx = x \operatorname{ch} x - \operatorname{sh} x + C. \end{aligned}$$

$$1801. \int x^3 \operatorname{ch} 3x dx.$$

$$\begin{aligned} \text{解} \quad \int x^3 \operatorname{ch} 3x dx &= \frac{1}{3} \int x^3 d(\operatorname{sh} 3x) \\ &= \frac{1}{3} x^3 \operatorname{sh} 3x - \int x^2 \operatorname{sh} 3x dx \\ &= \frac{1}{3} x^3 \operatorname{sh} 3x - \frac{1}{3} \int x^2 d(\operatorname{ch} 3x) \\ &= \frac{1}{3} x^3 \operatorname{sh} 3x - \frac{1}{3} x^2 \operatorname{ch} 3x + \frac{2}{3} \int x \operatorname{ch} 3x dx \\ &= \frac{1}{3} x^3 \operatorname{sh} 3x - \frac{1}{3} x^2 \operatorname{ch} 3x + \frac{2}{9} \int x d(\operatorname{sh} 3x) \\ &= \frac{1}{3} x^3 \operatorname{sh} 3x - \frac{1}{3} x^2 \operatorname{ch} 3x + \frac{2}{9} x \operatorname{sh} 3x - \frac{2}{9} \int \operatorname{sh} 3x dx \\ &= \left(\frac{x^3}{3} + \frac{2x}{9} \right) \operatorname{sh} 3x - \left(\frac{x^2}{3} + \frac{2}{27} \right) \operatorname{ch} 3x + C. \end{aligned}$$

$$1802. \int \operatorname{arc} \operatorname{tg} x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{arc} \operatorname{tg} x dx &= x \operatorname{arc} \operatorname{tg} x - \int \frac{x}{1+x^2} dx \\ &= x \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \ln(1+x^2) + C. \end{aligned}$$

$$1803. \int \operatorname{arc} \sin x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{arc} \sin x dx &= x \operatorname{arc} \sin x - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \operatorname{arc} \sin x + \sqrt{1-x^2} + C. \end{aligned}$$

$$1804. \int x \operatorname{arc} \operatorname{tg} x dx.$$

$$\begin{aligned} \text{解} \quad \int x \operatorname{arc} \operatorname{tg} x dx &= \frac{1}{2} \int \operatorname{arc} \operatorname{tg} x d(x^2) \\ &= \frac{1}{2} x^2 \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{1}{2} x^2 \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{1+x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{x}{2} + C. \end{aligned}$$

$$1805. \int x^2 \operatorname{arc} \cos x dx.$$

$$\begin{aligned} \text{解} \quad \int x^2 \operatorname{arc} \cos x dx &= \frac{1}{3} \int \operatorname{arc} \cos x d(x^3) \\ &= \frac{1}{3} x^3 \operatorname{arc} \cos x + \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx \\ &= \frac{1}{3} x^3 \operatorname{arc} \cos x - \frac{1}{6} \int \frac{x^2}{\sqrt{1-x^2}} d(1-x^2) \\ &= \frac{1}{3} x^3 \operatorname{arc} \cos x - \frac{1}{6} \int \left(\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2} \right) d(1-x^2) \\ &= \frac{1}{3} x^3 \operatorname{arc} \cos x - \frac{1}{3} \sqrt{1-x^2} + \frac{1}{9} (1-x^2)^{\frac{3}{2}} + C \\ &= \frac{1}{3} x^3 \operatorname{arc} \cos x - \frac{x^2+2}{9} \sqrt{1-x^2} + C. \end{aligned}$$

$$1806. \int \frac{\operatorname{arc} \sin x}{x^2} dx.$$

$$\text{解 } \int \frac{\arcsin x}{x^2} dx = - \int \arcsin x d\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x} \arcsin x + \int \frac{dx}{x\sqrt{1-x^2}}$$

作代换 $x = \sin t$, 得

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos t dt}{\sin t \cos t} = \int \frac{dt}{\sin t} = \ln \left| \operatorname{tg} \frac{t}{2} \right| + C^*$$

$$= \ln \left| \frac{\sin t}{1 + \cos t} \right| + C = -\ln \left| \frac{1 + \cos t}{\sin t} \right| + C$$

$$= -\ln \left| \frac{1 + \sqrt{1-x^2}}{x} \right| + C,$$

最后得

$$\int \frac{\arcsin x}{x^2} dx = -\frac{1}{x} \arcsin x$$

$$- \ln \left| \frac{1 + \sqrt{1-x^2}}{x} \right| + C.$$

*) 利用1703题的结果。

$$1807. \int \ln(x + \sqrt{1+x^2}) dx.$$

$$\text{解 } \int \ln(x + \sqrt{1+x^2}) dx$$

$$= x \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx$$

$$= x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

$$1808. \int x \ln \frac{1+x}{1-x} dx.$$

$$\begin{aligned} \text{解} \quad \int x \ln \frac{1+x}{1-x} dx &= \frac{1}{2} \int \ln \frac{1+x}{1-x} d(x^2) \\ &= \frac{x^2}{2} \ln \frac{1+x}{1-x} - \int \frac{x^2}{1-x^2} dx \\ &= \frac{x^2}{2} \ln \frac{1+x}{1-x} + \int \left(1 - \frac{1}{1-x^2} \right) dx \\ &= x - \frac{1-x^2}{2} \ln \frac{1+x}{1-x} + C. \end{aligned}$$

$$1809. \int \operatorname{arc} \operatorname{tg} \sqrt{x} dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{arc} \operatorname{tg} \sqrt{x} dx &= x \operatorname{arc} \operatorname{tg} \sqrt{x} \\ &\quad - \frac{1}{2} \int \frac{x}{\sqrt{x}(1+x)} dx \\ &= x \operatorname{arc} \operatorname{tg} \sqrt{x} - \int \left(1 - \frac{1}{1+x} \right) d(\sqrt{x}) \\ &= (x+1) \operatorname{arc} \operatorname{tg} \sqrt{x} - \sqrt{x} + C. \end{aligned}$$

$$1810. \int \sin x \cdot \ln(\operatorname{tg} x) dx.$$

$$\begin{aligned} \text{解} \quad \int \sin x \cdot \ln(\operatorname{tg} x) dx &= - \int \ln(\operatorname{tg} x) d(\cos x) \\ &= - \cos x \cdot \ln(\operatorname{tg} x) + \int \cos x \cdot \operatorname{ctg} x \cdot \sec^2 x dx \\ &= - \cos x \cdot \ln(\operatorname{tg} x) + \int \frac{dx}{\sin x} \end{aligned}$$

$$= -\cos x \cdot \ln(\operatorname{tg} x) + \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.$$

求下列积分:

1811. $\int x^5 e^{x^3} dx.$

$$\begin{aligned} \text{解} \quad \int x^5 e^{x^3} dx &= \frac{1}{3} \int x^3 d(e^{x^3}) \\ &= \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} \int e^{x^3} d(x^3) \\ &= \frac{1}{3} (x^3 - 1) e^{x^3} + C. \end{aligned}$$

1812. $\int (\operatorname{arc} \sin x)^2 dx.$

$$\begin{aligned} \text{解} \quad \int (\operatorname{arc} \sin x)^2 dx &= x(\operatorname{arc} \sin x)^2 - 2 \int \frac{x \operatorname{arc} \sin x}{\sqrt{1-x^2}} dx \\ &= x(\operatorname{arc} \sin x)^2 + 2 \int \operatorname{arc} \sin x d(\sqrt{1-x^2}) \\ &= x(\operatorname{arc} \sin x)^2 + 2\sqrt{1-x^2} \operatorname{arc} \sin x - 2 \int dx \\ &= x(\operatorname{arc} \sin x)^2 + 2\sqrt{1-x^2} \operatorname{arc} \sin x - 2x + C. \end{aligned}$$

1813. $\int x(\operatorname{arc} \operatorname{tg} x)^2 dx.$

$$\begin{aligned} \text{解} \quad \int x(\operatorname{arc} \operatorname{tg} x)^2 dx &= \frac{1}{2} \int (\operatorname{arc} \operatorname{tg} x)^2 d(x^2) \\ &= \frac{1}{2} x^2 (\operatorname{arc} \operatorname{tg} x)^2 - \int \frac{x^2 \operatorname{arc} \operatorname{tg} x}{1+x^2} dx \\ &= \frac{x^2}{2} (\operatorname{arc} \operatorname{tg} x)^2 - \int \left(1 - \frac{1}{1+x^2} \right) \operatorname{arc} \operatorname{tg} x dx \end{aligned}$$

$$\begin{aligned}
&= \frac{x^2}{2} (\operatorname{arc} \operatorname{tg} x)^2 - \int \operatorname{arc} \operatorname{tg} x dx + \int \operatorname{arc} \operatorname{tg} x d(\operatorname{arc} \operatorname{tg} x) \\
&= \frac{x^2}{2} (\operatorname{arc} \operatorname{tg} x)^2 - x \operatorname{arc} \operatorname{tg} x + \int \frac{x dx}{1+x^2} + \frac{1}{2} (\operatorname{arc} \operatorname{tg} x)^2 \\
&= \frac{x^2+1}{2} (\operatorname{arc} \operatorname{tg} x)^2 - x \operatorname{arc} \operatorname{tg} x + \frac{1}{2} \ln(1+x^2) + C.
\end{aligned}$$

1814. $\int x^2 \ln \frac{1-x}{1+x} dx.$

解 $\int x^2 \ln \frac{1-x}{1+x} dx = \frac{1}{3} \int \ln \frac{1-x}{1+x} d(x^3)$

$$\begin{aligned}
&= \frac{1}{3} x^3 \ln \frac{1-x}{1+x} + \frac{2}{3} \int \frac{x^3}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1-x}{1+x} + \frac{2}{3} \int \left(-x + \frac{x}{1-x^2} \right) dx \\
&= \frac{x^3}{3} \ln \frac{1-x}{1+x} - \frac{1}{3} x^2 - \frac{1}{3} \ln(1-x^2) + C.
\end{aligned}$$

1815. $\int \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx.$

解 $\int \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$

$$\begin{aligned}
&= \int \ln(x + \sqrt{1+x^2}) d(\sqrt{1+x^2}) \\
&= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) - \int \sqrt{1+x^2} \cdot \frac{1}{\sqrt{1+x^2}} dx \\
&= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) - x + C.
\end{aligned}$$

$$1816. \int \frac{x^2}{(1+x^2)^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^2}{(1+x^2)^2} dx &= \frac{1}{2} \int \frac{x}{(1+x^2)^2} d(1+x^2) \\ &= -\frac{1}{2} \int x d\left(\frac{1}{1+x^2}\right) = -\frac{x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{1+x^2} \\ &= -\frac{x}{2(1+x^2)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} x + C. \end{aligned}$$

$$1817. \int \frac{dx}{(a^2+x^2)^2}$$

解 当 $a = 0$ 时,

$$\int \frac{dx}{(a^2+x^2)^2} = \int \frac{dx}{x^4} = -\frac{1}{3x^3} + C,$$

当 $a \neq 0$ 时,

$$\begin{aligned} \int \frac{dx}{(a^2+x^2)^2} &= \frac{1}{a^2} \int \frac{(a^2+x^2) - x^2}{(a^2+x^2)^2} dx \\ &= \frac{1}{a^2} \int \frac{dx}{a^2+x^2} - \frac{1}{a^2} \int \frac{x^2}{(a^2+x^2)^2} dx \\ &= \frac{1}{a^3} \operatorname{arc} \operatorname{tg} \frac{x}{a} - \frac{1}{a^3} \int \frac{\left(\frac{x}{a}\right)^2 d\left(\frac{x}{a}\right)}{\left[1+\left(\frac{x}{a}\right)^2\right]^2} \\ &= \frac{1}{a^3} \operatorname{arc} \operatorname{tg} \frac{x}{a} - \frac{1}{a^3} \left[-\frac{\frac{x}{a}}{2\left(1+\frac{x^2}{a^2}\right)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} \frac{x}{a} \right] + C \end{aligned}$$

$$= \frac{1}{2a^3} \operatorname{arctg} \frac{x}{a} + \frac{x}{2a^2(a^2+x^2)} + C.$$

*) 利用1816题的结果.

$$1818. \int \sqrt{a^2-x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{a^2-x^2} dx &= x \sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}} dx \\ &= x \sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}} dx \\ &= x \sqrt{a^2-x^2} + a^2 \operatorname{arcsin} \frac{x}{|a|} - \int \sqrt{a^2-x^2} dx, \end{aligned}$$

于是得

$$\begin{aligned} \int \sqrt{a^2-x^2} dx &= \frac{x}{2} \sqrt{a^2-x^2} \\ &+ \frac{a^2}{2} \operatorname{arcsin} \frac{x}{|a|} + C \quad (a \neq 0). \end{aligned}$$

$$1819. \int \sqrt{x^2+a} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{x^2+a} dx &= x \sqrt{x^2+a} - \int \frac{x^2 dx}{\sqrt{x^2+a}} \\ &= x \sqrt{x^2+a} - \int \sqrt{x^2+a} dx + a \int \frac{dx}{\sqrt{x^2+a}}, \end{aligned}$$

于是得

$$\int \sqrt{x^2+a} dx = \frac{x}{2} \sqrt{x^2+a} + \frac{a}{2} \int \frac{dx}{\sqrt{x^2+a}}$$

$$= \frac{x}{2} \sqrt{x^2 + a} + \frac{a}{2} \ln |x + \sqrt{x^2 + a}| + C.$$

1820. $\int x^2 \sqrt{a^2 + x^2} dx.$

$$\begin{aligned} \text{解} \quad \int x^2 \sqrt{a^2 + x^2} dx &= \frac{1}{2} \int x (a^2 + x^2)^{\frac{1}{2}} d(a^2 + x^2) \\ &= \frac{1}{3} \int x d \left[(a^2 + x^2)^{\frac{3}{2}} \right] \\ &= \frac{1}{3} x (a^2 + x^2)^{\frac{3}{2}} - \frac{1}{3} \int (a^2 + x^2)^{\frac{3}{2}} dx \\ &= \frac{1}{3} x (a^2 + x^2) \sqrt{a^2 + x^2} - \frac{a^2}{3} \int \sqrt{a^2 + x^2} dx \\ &\quad - \frac{1}{3} \int x^2 \sqrt{a^2 + x^2} dx. \end{aligned}$$

于是得

$$\begin{aligned} \int x^2 \sqrt{a^2 + x^2} dx &= \frac{3}{4} \left[\frac{1}{3} x (a^2 + x^2) \sqrt{a^2 + x^2} \right. \\ &\quad \left. - \frac{a^2}{3} \int \sqrt{a^2 + x^2} dx \right] \\ &= \frac{1}{4} x (a^2 + x^2) \sqrt{a^2 + x^2} \\ &\quad - \frac{a^2}{4} \left[\frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) \right]^* + C \\ &= \frac{x(2x^2 + a^2)}{8} \sqrt{a^2 + x^2} - \frac{a^4}{8} \ln(x + \sqrt{x^2 + a^2}) + C. \end{aligned}$$

*) 利用1786题的结果.

$$1821. \int x \sin^2 x dx.$$

$$\begin{aligned} \text{解} \quad \int x \sin^2 x dx &= \frac{1}{2} \int x(1 - \cos 2x) dx \\ &= \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx \\ &= \frac{1}{4} x^2 - \frac{1}{4} \int x d(\sin 2x) \\ &= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x + \frac{1}{4} \int \sin 2x dx \\ &= \frac{1}{4} x^2 - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C. \end{aligned}$$

$$1822. \int e^{\sqrt{x}} dx.$$

解 设 $\sqrt{x} = t$, 则 $x = t^2$, $dx = 2t dt$, 代入得

$$\begin{aligned} \int e^{\sqrt{x}} dx &= 2 \int t e^t dt = 2 \int t d(e^t) \\ &= 2te^t - 2 \int e^t dt \\ &= 2te^t - 2e^t + C = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C. \end{aligned}$$

$$1823. \int x \sin \sqrt{x} dx.$$

解 设 $\sqrt{x} = t$, 则 $x = t^2$, $dx = 2t dt$, 代入得

$$\int x \sin \sqrt{x} dx = 2 \int t^3 \sin t dt = -2 \int t^3 d(\cos t)$$

$$\begin{aligned}
&= -2t^3 \cos t + 6 \int t^2 \cos t dt \\
&= -2t^3 \cos t + 6 \int t^2 d(\sin t) \\
&= -2t^3 \cos t + 6t^2 \sin t - 12 \int t \sin t dt \\
&= -2t^3 \cos t + 6t^2 \sin t + 12 \int t d(\cos t) \\
&= -2t^3 \cos t + 6t^2 \sin t + 12t \cos t - 12 \int \cos t dt \\
&= -2(t^2 - 6)t \cos t + 6(t^2 - 2) \sin t + C \\
&= 2(6 - x)\sqrt{x} \cos \sqrt{x} - 6(2 - x) \sin \sqrt{x} + C.
\end{aligned}$$

1824. $\int \frac{x e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx.$

解 $\int \frac{x e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx = \int \frac{x}{\sqrt{1+x^2}} d(e^{\arctg x})$

$$\begin{aligned}
&= \frac{x}{\sqrt{1+x^2}} e^{\arctg x} - \int e^{\arctg x} \cdot \frac{1}{(1+x^2)^{\frac{3}{2}}} dx \\
&= \frac{x}{\sqrt{1+x^2}} e^{\arctg x} - \int \frac{1}{\sqrt{1+x^2}} d(e^{\arctg x}) \\
&= \frac{x-1}{\sqrt{1+x^2}} e^{\arctg x} - \int \frac{x e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx,
\end{aligned}$$

于是得

$$\int \frac{x e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx = \frac{x-1}{2\sqrt{1+x^2}} e^{\arctg x} + C.$$

$$1825. \int \frac{e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx.$$

$$\text{解} \quad \int \frac{e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx = \int \frac{1}{\sqrt{1+x^2}} d(e^{\arctg x})$$

$$= \frac{1}{\sqrt{1+x^2}} e^{\arctg x} + \int \frac{x e^{\arctg x}}{(1+x^2)^{\frac{3}{2}}} dx$$

$$= \frac{1}{\sqrt{1+x^2}} e^{\arctg x} + \frac{x-1}{2\sqrt{1+x^2}} e^{\arctg x} + C$$

$$= \frac{x+1}{2\sqrt{1+x^2}} e^{\arctg x} + C.$$

*) 利用1824题的结果.

$$1826. \int \sin(\ln x) dx.$$

$$\text{解} \quad \int \sin(\ln x) dx = x \sin(\ln x) - \int x \cos(\ln x) \cdot \frac{1}{x} dx$$

$$= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx,$$

于是得

$$\int \sin(\ln x) dx = \frac{x}{2} (\sin(\ln x) - \cos(\ln x)) + C.$$

$$1827. \int \cos(\ln x) dx.$$

$$\text{解} \quad \int \cos(\ln x) dx = x \cos(\ln x) + \int \sin(\ln x) dx$$

$$= x \cos(\ln x) + \frac{x}{2} (\sin(\ln x) - \cos(\ln x)) + C$$

$$= \frac{x}{2} (\sin(\ln x) + \cos(\ln x)) + C.$$

*) 利用1826题的结果.

1828. $\int e^{ax} \cos bx dx.$

解 如果 a, b 同时为零, 积分显然为 $x + C$; 若 $a = 0$,
 $b \neq 0$, 积分显然为 $\frac{1}{b} \sin bx + C$; 以下设 $a \neq 0$:

$$\begin{aligned} \int e^{ax} \cos bx dx &= \frac{1}{a} \int \cos bx d(e^{ax}) \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx dx \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \int \sin bx d(e^{ax}) \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx dx, \end{aligned}$$

于是得

$$\begin{aligned} \int e^{ax} \cos bx dx &= \frac{a^2}{a^2 + b^2} \left[\frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx \right] \\ &+ C = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} + C. \end{aligned}$$

1829. $\int e^{ax} \sin bx dx.$

解 若 $a = b = 0$, 则积分为 $x + C$; 以下设 $a^2 + b^2 \neq 0$, 则有

$$\int e^{ax} \sin bx dx = \frac{1}{a} \int \sin bx d(e^{ax})$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} \int \cos bx de^{ax}$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx dx,$$

$$\text{故 } \int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$$

$$1830. \int e^{2x} \sin^2 x dx.$$

$$\text{解 } \int e^{2x} \sin^2 x dx = \frac{1}{2} \int e^{2x} (1 - \cos 2x) dx$$

$$= \frac{1}{2} \int e^{2x} dx - \frac{1}{2} \int e^{2x} \cos 2x dx$$

$$= \frac{1}{4} e^{2x} - \frac{1}{2} \left(\frac{2 \cos 2x + 2 \sin 2x}{8} e^{2x} \right)^{*}) + C$$

$$= \frac{1}{8} e^{2x} (2 - \cos 2x - \sin 2x) + C.$$

*) 利用1828题的结果。

$$1831. \int (e^x - \cos x)^2 dx.$$

$$\text{解 } \int (e^x - \cos x)^2 dx = \int (e^{2x} - 2e^x \cos x + \cos^2 x) dx$$

$$= \frac{1}{2} e^{2x} - 2 \cdot \frac{e^x (\cos x + \sin x)^{*})}{2} + \left(\frac{x}{2} + \frac{1}{4} \sin 2x \right)^{**}) + C$$

$$= \frac{1}{2}e^{2x} - e^x(\cos x + \sin x) + \frac{x}{2} + \frac{1}{4}\sin 2x + C.$$

*) 利用1828题的结果.

***) 利用1742题的结果.

$$1832. \int \frac{\operatorname{arc\,ctg} e^x}{e^x} dx.$$

$$\text{解} \quad \int \frac{\operatorname{arc\,ctg} e^x}{e^x} dx = - \int \operatorname{arc\,ctg} e^x d(e^{-x})$$

$$= -e^{-x} \cdot \operatorname{arc\,ctg}(e^x) - \int \frac{dx}{1+e^{2x}}$$

$$= -e^{-x} \operatorname{arc\,ctg}(e^x) + \frac{1}{2} [2x - \ln(1+e^{2x})] + C^*)$$

$$= -e^{-x} \operatorname{arc\,ctg}(e^x) - x + \frac{1}{2} \ln(1+e^{2x}) + C.$$

*) 利用1759题的结果.

$$1833. \int \frac{\ln(\sin x)}{\sin^2 x} dx.$$

$$\text{解} \quad \int \frac{\ln(\sin x)}{\sin^2 x} dx = - \int \ln(\sin x) d(\operatorname{ctg} x)$$

$$= -\operatorname{ctg} x \cdot \ln(\sin x) + \int \operatorname{ctg}^2 x dx$$

$$= -\operatorname{ctg} x \cdot \ln(\sin x) + (-\operatorname{ctg} x - x) + C^*)$$

$$= -[x + \operatorname{ctg} x \cdot \ln(\sin x)] + C.$$

*) 利用1649题或1751题的结果.

$$1834. \int \frac{x dx}{\cos^2 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x dx}{\cos^2 x} &= \int x d(\operatorname{tg} x) = x \operatorname{tg} x - \int \operatorname{tg} x dx \\ &= x \operatorname{tg} x + \ln |\cos x| + C. \end{aligned}$$

*) 利用1697题的结果.

$$1835. \int \frac{x e^x}{(x+1)^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x e^x}{(x+1)^2} dx &= - \int x e^x d\left(\frac{1}{1+x}\right) \\ &= - \frac{x}{1+x} e^x + \int \frac{1}{1+x} e^x (x+1) dx \\ &= - \frac{x}{1+x} e^x + e^x + C = \frac{e^x}{1+x} + C. \end{aligned}$$

下列积分的求法需要把二次三项式化成正则型, 并利用下列公式:

$$\text{I.} \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arc} \operatorname{tg} \frac{x}{a} + C \quad (a \neq 0);$$

$$\text{II.} \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad (a \neq 0);$$

$$\text{III.} \quad \int \frac{x dx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln |a^2 \pm x^2| + C;$$

$$\text{IV.} \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname{arc} \sin \frac{x}{a} + C \quad (a > 0);$$

$$\text{V.} \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C;$$

$$\text{VI. } \int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2} + C,$$

$$\text{VII. } \int \sqrt{a^2 - x^2} dx$$

$$= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C \quad (a > 0),$$

$$\text{VIII. } \int \sqrt{x^2 \pm a^2} dx$$

$$= \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln |x + \sqrt{x^2 \pm a^2}| + C.$$

求下列积分:

$$1836^+. \int \frac{dx}{a + bx^2} \quad (ab \neq 0).$$

解 当 $ab > 0$ 时,

$$\int \frac{dx}{a + bx^2} = \operatorname{sgn} a \cdot \frac{1}{\sqrt{|b|}} \int \frac{d(\sqrt{|b|x})}{(\sqrt{|a|})^2 + (\sqrt{|b|x})^2}$$

$$= \operatorname{sgn} a \cdot \frac{1}{\sqrt{ab}} \arctg \left(x \sqrt{\frac{b}{a}} \right) + C,$$

当 $ab < 0$ 时,

$$\int \frac{dx}{a + bx^2} = \operatorname{sgn} a \cdot \int \frac{dx}{|a| - |b|x^2}$$

$$= \operatorname{sgn} a \cdot \frac{1}{\sqrt{|b|}} \int \frac{d(\sqrt{|b|x})}{(\sqrt{|a|})^2 - (\sqrt{|b|x})^2}$$

$$= \frac{\operatorname{sgn} a}{2\sqrt{-ab}} \ln \left| \frac{\sqrt{|a|} + x\sqrt{|b|}}{\sqrt{|a|} - x\sqrt{|b|}} \right| + C.$$

1837. $\int \frac{dx}{x^2 - x + 2}.$

解 $\int \frac{dx}{x^2 - x + 2} = \int \frac{d(x - \frac{1}{2})}{(x - \frac{1}{2})^2 + (\frac{\sqrt{7}}{2})^2}$

$$= \frac{2}{\sqrt{7}} \operatorname{arc} \operatorname{tg} \frac{2x - 1}{\sqrt{7}} + C.$$

1838. $\int \frac{dx}{3x^2 - 2x - 1}.$

解 $\int \frac{dx}{3x^2 - 2x - 1} = \frac{1}{3} \int \frac{dx}{x^2 - \frac{2}{3}x - \frac{1}{3}}$

$$= \frac{1}{3} \int \frac{d(x - \frac{1}{3})}{(x - \frac{1}{3})^2 - (\frac{2}{3})^2}$$

$$= -\frac{1}{3} \cdot \frac{3}{4} \ln \left| \frac{\frac{2}{3} + (x - \frac{1}{3})}{\frac{2}{3} - (x - \frac{1}{3})} \right| + C_1$$

$$= \frac{1}{4} \ln \left| \frac{x - 1}{3x + 1} \right| + C.$$

1839. $\int \frac{x dx}{x^4 - 2x^2 - 1}.$

解 $\int \frac{x dx}{x^4 - 2x^2 - 1} = \frac{1}{2} \int \frac{d(x^2 - 1)}{(x^2 - 1)^2 - (\sqrt{2})^2}$

$$= \frac{1}{4\sqrt{2}} \ln \left| \frac{x^2 - (\sqrt{2} + 1)}{x^2 + (\sqrt{2} - 1)} \right| + C.$$

1840. $\int \frac{x+1}{x^2+x+1} dx.$

解 $\int \frac{x+1}{x^2+x+1} dx = \int \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1} dx$

$$= \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2} \ln(x^2+x+1) + \frac{1}{\sqrt{3}} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C.$$

1841. $\int \frac{x dx}{x^2 - 2x \cos \alpha + 1}.$

解 $\int \frac{x dx}{x^2 - 2x \cos \alpha + 1} = \int \frac{x - \cos \alpha + \cos \alpha}{(x - \cos \alpha)^2 + \sin^2 \alpha} dx$

$$= \frac{1}{2} \int \frac{d[(x - \cos \alpha)^2 + \sin^2 \alpha]}{(x - \cos \alpha)^2 + \sin^2 \alpha}$$

$$+ \cos \alpha \cdot \int \frac{d(x - \cos \alpha)}{(x - \cos \alpha)^2 + \sin^2 \alpha}$$

$$= \frac{1}{2} \ln(x^2 - 2x \cos \alpha + 1) + \operatorname{ctg} \alpha \cdot \operatorname{arctg} \left(\frac{x - \cos \alpha}{\sin \alpha} \right) + C$$

$$(\alpha \neq k\pi, k=0, \pm 1, \pm 2, \dots).$$

1842. $\int \frac{x^3 dx}{x^4 - x^2 + 2}.$

$$\begin{aligned}
\text{解} \quad \int \frac{x^3 dx}{x^4 - x^2 + 2} &= \frac{1}{2} \int \frac{x^2 d(x^2)}{\left(x^2 - \frac{1}{2}\right)^2 + \frac{7}{4}} \\
&= \frac{1}{2} \int \frac{\left(x^2 - \frac{1}{2}\right) + \frac{1}{2}}{\left(x^2 - \frac{1}{2}\right)^2 + \frac{7}{4}} d\left(x^2 - \frac{1}{2}\right) \\
&= \frac{1}{4} \int \frac{d\left(x^2 - \frac{1}{2}\right)^2}{\left(x^2 - \frac{1}{2}\right)^2 + \frac{7}{4}} + \frac{1}{4} \int \frac{d\left(x^2 - \frac{1}{2}\right)}{\left(x^2 - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\
&= \frac{1}{4} \ln(x^4 - x^2 + 2) + \frac{1}{2\sqrt{7}} \operatorname{arc} \operatorname{tg}\left(\frac{2x^2 - 1}{\sqrt{7}}\right) + C.
\end{aligned}$$

1243. $\int \frac{x^5 dx}{x^6 - x^3 - 2}$

$$\begin{aligned}
\text{解} \quad \int \frac{x^5 dx}{x^6 - x^3 - 2} &= \frac{1}{3} \int \frac{x^3 d(x^3)}{\left(x^3 - \frac{1}{2}\right)^2 - \frac{9}{4}} \\
&= \frac{1}{3} \int \frac{\left(x^3 - \frac{1}{2}\right) + \frac{1}{2}}{\left(x^3 - \frac{1}{2}\right)^2 - \frac{9}{4}} d\left(x^3 - \frac{1}{2}\right) \\
&= \frac{1}{6} \int \frac{d\left(x^3 - \frac{1}{2}\right)^2}{\left(x^3 - \frac{1}{2}\right)^2 - \frac{9}{4}} - \frac{1}{6} \int \frac{d\left(x^3 - \frac{1}{2}\right)}{\left(\frac{3}{2}\right)^2 - \left(x^3 - \frac{1}{2}\right)^2} \\
&= \frac{1}{6} \ln|x^6 - x^3 - 2| - \frac{1}{18} \ln \left| \frac{\frac{3}{2} + \left(x^3 - \frac{1}{2}\right)}{\frac{3}{2} - \left(x^3 - \frac{1}{2}\right)} \right| + C
\end{aligned}$$

$$= \frac{1}{9} \ln \{ |x^3 + 1| \cdot (x^3 - 2)^2 \} + C.$$

如果本题不化成正则型来作，则有更简单的作法，事实上，

$$\begin{aligned} \int \frac{x^5 dx}{x^6 - x^3 - 2} &= \frac{1}{3} \int \frac{x^3 d(x^3)}{(x^3 - 2)(x^3 + 1)} \\ &= \frac{1}{9} \int \left(\frac{2}{x^3 - 2} + \frac{1}{x^3 + 1} \right) d(x^3) \\ &= \frac{1}{9} \ln \{ |x^3 + 1| \cdot (x^3 - 2)^2 \} + C. \end{aligned}$$

1844. $\int \frac{dx}{3\sin^2 x - 8\sin x \cos x + 5\cos^2 x}$

$$\begin{aligned} \text{解} \quad & \int \frac{dx}{3\sin^2 x - 8\sin x \cos x + 5\cos^2 x} \\ &= \int \frac{d(\operatorname{tg} x)}{3\operatorname{tg}^2 x - 8\operatorname{tg} x + 5} \\ &= \frac{1}{3} \int \frac{d\left(\operatorname{tg} x - \frac{4}{3}\right)}{\left(\operatorname{tg} x - \frac{4}{3}\right)^2 - \left(\frac{1}{3}\right)^2} \\ &= \frac{1}{2} \ln \left| \frac{\frac{1}{3} - \left(\operatorname{tg} x - \frac{4}{3}\right)}{\frac{1}{3} + \left(\operatorname{tg} x - \frac{4}{3}\right)} \right| + C_1 \\ &= \frac{1}{2} \ln \left| \frac{3\sin x + 5\cos x}{\sin x - \cos x} \right| + C. \end{aligned}$$

$$1845. \int \frac{dx}{\sin x + 2\cos x + 3}$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin x + 2\cos x + 3} &= \int \frac{\frac{1}{\cos^2 \frac{x}{2}} dx}{2 \operatorname{tg} \frac{x}{2} + 4 + \operatorname{sec}^2 \frac{x}{2}} \\ &= 2 \int \frac{d\left(\operatorname{tg} \frac{x}{2}\right)}{\left(\operatorname{tg} \frac{x}{2} + 1\right)^2 + 4} = \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} \frac{x}{2} + 1}{2}\right) + C. \end{aligned}$$

$$1846. \int \frac{dx}{\sqrt{a+bx^2}} \quad (b \neq 0).$$

解 当 $b > 0$ 时,

$$\int \frac{dx}{\sqrt{a+bx^2}} = \frac{1}{\sqrt{b}} \ln \left| x\sqrt{b} + \sqrt{a+bx^2} \right| + C;$$

当 $a > 0$ 及 $b < 0$ 时,

$$\begin{aligned} \int \frac{dx}{\sqrt{a+bx^2}} &= \frac{1}{\sqrt{-b}} \int \frac{d(\sqrt{-bx})}{\sqrt{(\sqrt{a})^2 - (\sqrt{-bx})^2}} \\ &= \frac{1}{\sqrt{-b}} \operatorname{arc} \sin \left(x\sqrt{-\frac{b}{a}} \right) + C. \end{aligned}$$

$$1847. \int \frac{dx}{\sqrt{1-2x-x^2}}$$

$$\text{解} \quad \int \frac{dx}{\sqrt{1-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{2-(x+1)^2}}$$

$$= \arcsin\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

1848. $\int \frac{dx}{\sqrt{x+x^2}}.$

解 $\int \frac{dx}{\sqrt{x+x^2}} = \int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}}$

$$= \ln\left|x+\frac{1}{2} + \sqrt{x+x^2}\right| + C.$$

本题即1687题，注意不同的解法及不同形式的结果。

1849. $\int \frac{dx}{\sqrt{2x^2-x+2}}.$

解 $\int \frac{dx}{\sqrt{2x^2-x+2}} = \frac{1}{\sqrt{2}} \int \frac{d\left(x-\frac{1}{4}\right)}{\sqrt{\left(x-\frac{1}{4}\right)^2 + \frac{15}{16}}}$

$$= \frac{1}{\sqrt{2}} \ln\left(x-\frac{1}{4} + \sqrt{x^2 - \frac{x}{2} + 1}\right) + C.$$

1850. 证明：若

$$y = ax^2 + bx + c \quad (a \neq 0),$$

则当 $a > 0$ 时， $\int \frac{dx}{\sqrt{y}} = \frac{1}{\sqrt{a}} \ln\left|\frac{y'}{2} + \sqrt{ay}\right| + C,$

当 $a < 0$ 时， $\int \frac{dx}{\sqrt{y}} = \frac{1}{\sqrt{-a}} \arcsin \frac{-y'}{\sqrt{b^2 - 4ac}} + C.$

证 当 $a > 0$ 时，

$$\begin{aligned}
\int \frac{dx}{\sqrt{y}} &= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}} \\
&= \frac{1}{\sqrt{a}} \int \frac{d\left(x + \frac{b}{2a}\right)}{\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}}} \\
&= \frac{1}{\sqrt{a}} \ln \left| x + \frac{b}{2a} + \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} \right| + C \\
&= \frac{1}{\sqrt{a}} \ln \left| \frac{y'}{2} + \sqrt{ay} \right| + C,
\end{aligned}$$

当 $a < 0$ 时,

$$\begin{aligned}
\int \frac{dx}{\sqrt{y}} &= \frac{1}{\sqrt{-a}} \int \frac{dx}{\sqrt{-x^2 - \frac{b}{a}x - \frac{c}{a}}} \\
&= \frac{1}{\sqrt{-a}} \int \frac{d\left(x + \frac{b}{2a}\right)}{\sqrt{\frac{b^2 - 4ac}{4a^2} - \left(x + \frac{b}{2a}\right)^2}} \\
&= \frac{1}{\sqrt{-a}} \arcsin \left(\frac{x + \frac{b}{2a}}{\frac{\sqrt{b^2 - 4ac}}{-2a}} \right) + C \\
&= \frac{1}{\sqrt{-a}} \arcsin \left(\frac{-y'}{\sqrt{b^2 - 4ac}} \right) + C.
\end{aligned}$$

1851. $\int \frac{x dx}{\sqrt{5+x-x^2}}$.

$$\begin{aligned}
 \text{解} \quad \int \frac{x dx}{\sqrt{5+x-x^2}} &= \int \frac{\left(x-\frac{1}{2}\right) + \frac{1}{2}}{\sqrt{\frac{21}{4} - \left(x-\frac{1}{2}\right)^2}} dx \\
 &= -\frac{1}{2} \int \frac{d\left[\frac{21}{4} - \left(x-\frac{1}{2}\right)^2\right]}{\sqrt{\frac{21}{4} - \left(x-\frac{1}{2}\right)^2}} + \frac{1}{2} \int \frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\frac{21}{4} - \left(x-\frac{1}{2}\right)^2}} \\
 &= -\sqrt{5+x-x^2} + \frac{1}{2} \arcsin\left(\frac{2x-1}{\sqrt{21}}\right) + C.
 \end{aligned}$$

$$1852. \int \frac{x+1}{\sqrt{x^2+x+1}} dx.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x+1}{\sqrt{x^2+x+1}} dx &= \int \frac{\left(x+\frac{1}{2}\right) + \frac{1}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} dx \\
 &= \frac{1}{2} \int \frac{d\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right]}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} + \frac{1}{2} \int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} \\
 &= \sqrt{x^2+x+1} + \frac{1}{2} \ln\left(x+\frac{1}{2} + \sqrt{x^2+x+1}\right) + C.
 \end{aligned}$$

$$1853. \int \frac{x dx}{\sqrt{1-3x^2-2x^4}}.$$

$$\begin{aligned}
 \text{解} \quad \int \frac{x dx}{\sqrt{1-3x^2-2x^4}} &= \frac{1}{2\sqrt{2}} \int \frac{d\left(x^2 + \frac{3}{4}\right)}{\sqrt{\frac{17}{16} - \left(x^2 + \frac{3}{4}\right)^2}} \\
 &= \frac{1}{2\sqrt{2}} \arcsin\left(\frac{4x^2+3}{\sqrt{17}}\right) + C.
 \end{aligned}$$

$$1854. \int \frac{x^3 dx}{\sqrt{x^4 - 2x^2 - 1}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3 dx}{\sqrt{x^4 - 2x^2 - 1}} &= \frac{1}{2} \int \frac{x^2 d(x^2)}{\sqrt{(x^2 - 1)^2 - 2}} \\ &= \frac{1}{2} \int \frac{(x^2 - 1) d(x^2 - 1)}{\sqrt{(x^2 - 1)^2 - 2}} + \frac{1}{2} \int \frac{d(x^2 - 1)}{\sqrt{(x^2 - 1)^2 - 2}} \\ &= \frac{1}{2} \sqrt{x^4 - 2x^2 - 1} \\ &\quad + \frac{1}{2} \ln \left| x^2 - 1 + \sqrt{x^4 - 2x^2 - 1} \right| + C. \end{aligned}$$

$$1855. \int \frac{x + x^3}{\sqrt{1 + x^2 - x^4}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x + x^3}{\sqrt{1 + x^2 - x^4}} dx &= \frac{1}{2} \int \frac{(1 + x^2) d(x^2)}{\sqrt{\frac{5}{4} - \left(x^2 - \frac{1}{2}\right)^2}} \\ &= \frac{1}{2} \int \frac{\left(x^2 - \frac{1}{2}\right) d\left(x^2 - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x^2 - \frac{1}{2}\right)^2}} + \frac{3}{4} \int \frac{d\left(x^2 - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x^2 - \frac{1}{2}\right)^2}} \\ &= -\frac{1}{2} \sqrt{1 + x^2 - x^4} + \frac{3}{4} \arcsin \left(\frac{2x^2 - 1}{\sqrt{5}} \right) + C. \end{aligned}$$

$$1856. \int \frac{dx}{x\sqrt{x^2 + x + 1}}.$$

$$\text{解} \quad \text{作代换 } t = \frac{1}{x}, \text{ 则 } x\sqrt{x^2 + x + 1} = \frac{\operatorname{sgn} t}{t^2} \sqrt{t^2 + t + 1},$$

$$dx = -\frac{dt}{t^2}, \text{ 于是}$$

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2+x+1}} &= -(\operatorname{sgn} t) \int \frac{dt}{\sqrt{t^2+t+1}} \\
&= -(\operatorname{sgn} t) \ln \left| t + \frac{1}{2} + \sqrt{t^2+t+1} \right|^{*}) + C_1 \\
&= -(\operatorname{sgn} x) \ln \left| \frac{x+2+2(\operatorname{sgn} x)\sqrt{x^2+x+1}}{2x} \right| + C_1.
\end{aligned}$$

故当 $x > 0$ 时,

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2+x+1}} \\
&= -\ln \left| \frac{x+2+2\sqrt{x^2+x+1}}{x} \right| + C.
\end{aligned}$$

当 $x < 0$ 时,

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2+x+1}} \\
&= -\ln \left| \frac{2x}{x+2-2\sqrt{x^2+x+1}} \right| + C_1 \\
&= -\ln \left| \frac{2x(x+2+2\sqrt{x^2+x+1})}{(x+2)^2-4(x^2+x+1)} \right| + C_1 \\
&= -\ln \left| \frac{x+2+2\sqrt{x^2+x+1}}{x} \right| + C.
\end{aligned}$$

总之, 不论 x 为正或为负, 均有

$$\int \frac{dx}{x\sqrt{x^2+x+1}} = -\ln \left| \frac{x+2+2\sqrt{x^2+x+1}}{x} \right| + C.$$

*) 利用1850题的结果.

$$1857. \int \frac{dx}{x^2 \sqrt{x^2 + x - 1}}.$$

解 作代换 $t = \frac{1}{x}$, 则 $x^2 \sqrt{x^2 + x - 1}$

$$= \operatorname{sgn} t \cdot \frac{\sqrt{-t^2 + t + 1}}{t^3}, \quad dx = -\frac{dt}{t^2}. \quad \text{于是}$$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + x - 1}} &= -(\operatorname{sgn} t) \int \frac{t}{\sqrt{-t^2 + t + 1}} dt \\ &= -(\operatorname{sgn} t) \cdot \left(-\frac{1}{2} \int \frac{d(-t^2 + t + 1)}{\sqrt{-t^2 + t + 1}} \right. \\ &\quad \left. + \frac{1}{2} \int \frac{dt}{\sqrt{-t^2 + t + 1}} \right) \\ &= -(\operatorname{sgn} t) \cdot \left(-\sqrt{-t^2 + t + 1} \right. \\ &\quad \left. + \frac{1}{2} \operatorname{arc} \sin \frac{2t-1}{\sqrt{5}} \right)^* + C \\ &= (\operatorname{sgn} x) \cdot \left[\frac{\sqrt{x^2 + x - 1}}{|x|} + \frac{1}{2} \operatorname{arc} \sin \left(\frac{x-2}{x\sqrt{5}} \right) \right] + C \\ &= \frac{\sqrt{x^2 + x - 1}}{x} + \frac{1}{2} \operatorname{arc} \sin \left(\frac{x-2}{|x|\sqrt{5}} \right) + C. \end{aligned}$$

其存在域为 $\left| x + \frac{1}{2} \right| \geq \frac{\sqrt{5}}{2}$.

*) 利用1850题的结果。

$$1858. \int \frac{dx}{(x+1) \sqrt{x^2 + 1}}.$$

解 设 $y = x + 1$, 本题即转化为1856题的类型. 由于解法类似, 且 $x + 1$ 的符号对结果没有影响, 故仅就 $x + 1 > 0$ 列出解法的主要步骤如下:

$$\begin{aligned} \int \frac{dx}{(x+1)\sqrt{x^2+1}} &= \int \frac{dy}{y\sqrt{y^2-2y+2}} \\ &= - \int \frac{d\left(\frac{1}{y}\right)}{\sqrt{\frac{2}{y^2}-\frac{2}{y}+1}} \\ &= -\frac{1}{\sqrt{2}} \ln \left| \frac{1}{y} - \frac{1}{2} + \frac{\sqrt{y^2-2y+2}}{y\sqrt{2}} \right| + C_1 \\ &= -\frac{1}{\sqrt{2}} \ln \left| \frac{1-x+\sqrt{2(x^2+1)}}{x+1} \right| + C. \end{aligned}$$

1859. $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$.

解 设 $x-1 = \frac{1}{t}$, 则

$$(x-1)\sqrt{x^2-2} = \frac{\sqrt{1+2t-t^2}}{t|t|}, \quad dx = -\frac{1}{t^2}dt,$$

代入得

$$\begin{aligned} \int \frac{dx}{(x-1)\sqrt{x^2-2}} &= - \int \frac{\operatorname{sgn}t dt}{\sqrt{1+2t-t^2}} \\ &= -\operatorname{sgn}t \cdot \arcsin\left(\frac{t-1}{\sqrt{2}}\right) + C \\ &= \arcsin\left(\frac{x-2}{|x-1|\sqrt{2}}\right) + C \quad (|x| > \sqrt{2}). \end{aligned}$$

$$1860^+ \cdot \int \frac{dx}{(x+2)^2 \sqrt{x^2+2x-5}}$$

解 设 $x+2 = \frac{1}{t}$, 则

$$(x+2)^2 \sqrt{x^2+2x-5} = \frac{\sqrt{1-2t-5t^2}}{t^2 |t|},$$

$$dx = -\frac{1}{t^2} dt,$$

代入得

$$\begin{aligned} \int \frac{dx}{(x+2)^2 \sqrt{x^2+2x-5}} &= - \int \frac{t \cdot \operatorname{sgn} t dt}{\sqrt{1-2t-5t^2}} \\ &= -\frac{1}{\sqrt{5}} \int \frac{\left[\left(t + \frac{1}{5} \right) - \frac{1}{5} \right] \operatorname{sgn} t dt}{\sqrt{\frac{6}{25} - \left(t + \frac{1}{5} \right)^2}} \\ &= \frac{1}{\sqrt{5}} \operatorname{sgn} t \cdot \sqrt{\frac{1}{5} - \frac{2}{5}t - t^2} \\ &\quad + \frac{1}{5\sqrt{5}} \operatorname{sgn} t \cdot \arcsin \left(\frac{5t+1}{\sqrt{6}} \right) + C \\ &= \frac{\sqrt{x^2+2x-5}}{5(x+2)} + \frac{1}{5\sqrt{5}} \arcsin \left(\frac{x+7}{|x+2|\sqrt{6}} \right) \\ &\quad + C. \end{aligned}$$

其存在域为满足不等式 $x^2+2x-5 > 0$ 的一切 x 值, 即 $|x+1| > \sqrt{6}$.

$$1861. \int \sqrt{2+x-x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{2+x-x^2} dx &= \int \sqrt{\frac{9}{4} - \left(x - \frac{1}{2}\right)^2} d\left(x - \frac{1}{2}\right) \\ &= \frac{2x-1}{4} \sqrt{2+x-x^2} + \frac{9}{8} \arcsin\left(\frac{2x-1}{3}\right) + C. \end{aligned}$$

$$1862. \quad \int \sqrt{2+x+x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{2+x+x^2} dx &= \int \sqrt{\frac{7}{4} + \left(x + \frac{1}{2}\right)^2} d\left(x + \frac{1}{2}\right) \\ &= \frac{2x+1}{4} \sqrt{2+x+x^2} + \frac{7}{8} \ln\left(x + \frac{1}{2} + \sqrt{2+x+x^2}\right) \\ &\quad + C. \end{aligned}$$

$$1863. \quad \int \sqrt{x^4 + 2x^2 - 1} x dx.$$

$$\begin{aligned} \text{解} \quad \int \sqrt{x^4 + 2x^2 - 1} x dx \\ &= \frac{1}{2} \int \sqrt{(x^2 + 1)^2 - 2} d(x^2 + 1) \\ &= \frac{x^2 + 1}{4} \sqrt{x^4 + 2x^2 - 1} \\ &\quad - \frac{1}{2} \ln\left(x^2 + 1 + \sqrt{x^4 + 2x^2 - 1}\right) + C. \end{aligned}$$

$$1864. \quad \int \frac{1-x+x^2}{x\sqrt{1+x-x^2}} dx.$$

解 由于

$$\int \frac{dx}{x\sqrt{1+x-x^2}} = -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right| + C_1$$

(可仿照1856题求得),

$$\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$= \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C_2,$$

$$\int \frac{x dx}{\sqrt{1+x-x^2}} = \int \frac{\left(x-\frac{1}{2}\right) + \frac{1}{2}}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}} d\left(x-\frac{1}{2}\right)$$

$$= -\sqrt{1+x-x^2} + \frac{1}{2} \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C_3,$$

所以

$$\int \frac{1-x+x^2}{x\sqrt{1+x-x^2}} dx = \int \frac{dx}{x\sqrt{1+x-x^2}}$$

$$= \int \frac{dx}{\sqrt{1+x-x^2}} + \int \frac{x dx}{\sqrt{1+x-x^2}}$$

$$= -\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{x}\right|$$

$$+ \frac{1}{2} \arcsin\left(\frac{1-2x}{\sqrt{5}}\right) - \sqrt{1+x-x^2} + C,$$

其中存在域为满足不等式 $1+x-x^2 > 0$ 且 $x \neq 0$ 的一切

x 值, 即 $\left|x-\frac{1}{2}\right| < \frac{\sqrt{5}}{2}$ 及 $x \neq 0$.

1865. $\int \frac{x^2+1}{x\sqrt{x^4+1}} dx.$

$$\begin{aligned}
 \text{解} \quad \int \frac{x^2+1}{x\sqrt{x^4+1}} dx &= \int \frac{\operatorname{sgn}x \cdot \left(1 + \frac{1}{x^2}\right)}{\sqrt{x^2 + \frac{1}{x^2}}} dx \\
 &= \int \frac{\operatorname{sgn}x d\left(x - \frac{1}{x}\right)}{\sqrt{\left(x - \frac{1}{x}\right)^2 + 2}} \\
 &= \operatorname{sgn}x \cdot \ln\left(x - \frac{1}{x} + \sqrt{\left(x - \frac{1}{x}\right)^2 + 2}\right) + C_1 \\
 &= \ln\left|\frac{x^2-1+\sqrt{x^4+1}}{x}\right| + C.
 \end{aligned}$$

§2. 有理函数的积分法

利用待定系数法，求下列积分。

1866. $\int \frac{2x+3}{(x-2)(x+5)} dx.$

解 设 $\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$ ，通分后应有

$$2x+3 = A(x+5) + B(x-2).$$

在这恒等式中，

令 $x=2^*$ ，得 $7=7A$ ， $A=1$ ；

令 $x=-5$ ，得 $-7=-7B$ ， $B=1$ 。

于是，

$$\int \frac{2x+3}{(x-2)(x+5)} dx = \int \left(\frac{1}{x-2} + \frac{1}{x+5}\right) dx$$

$$= \ln|(x-2)(x+5)| + C.$$

* 注意，这是一种习惯的说法。实际上，不能直接令 $x=2$ （因为上述恒等式是当 $x \neq 2$, $x \neq -5$ 时得出来的），而应令 $x \rightarrow 2$ 取极限，得 $7=7A$ ，以下类似情况都作此理解。

$$1867. \int \frac{x dx}{(x+1)(x+2)(x+3)}.$$

$$\text{解 设 } \frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3},$$

通分后应有

$$x \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2).$$

在这恒等式中，

$$\text{令 } x = -1, \text{ 得 } -1 = 2A, \quad A = -\frac{1}{2},$$

$$\text{令 } x = -2, \text{ 得 } -2 = -B, \quad B = 2;$$

$$\text{令 } x = -3, \text{ 得 } -3 = 2C, \quad C = -\frac{3}{2}.$$

于是，

$$\begin{aligned} & \int \frac{x dx}{(x+1)(x+2)(x+3)} \\ &= \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{2}{x+2} + \frac{-\frac{3}{2}}{x+3} \right) dx \\ &= -\frac{1}{2} \ln|x+1| + 2 \ln|x+2| - \frac{3}{2} \ln|x+3| + C \end{aligned}$$

$$= \frac{1}{2} \ln \left| \frac{(x+2)^4}{(x+1)(x+3)^3} \right| + C.$$

1868. $\int \frac{x^{10}}{x^2+x-2} dx.$

解 $\frac{x^{10}}{x^2+x-2} = x^8 - x^7 + 3x^6 - 5x^5 + 11x^4 - 21x^3$
 $+ 43x^2 - 85x + 171 + \frac{-341x+342}{x^2+x-2},$

设 $\frac{-341x+342}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$, 通分后应有

$$-341x+342 \equiv A(x-1) + B(x+2),$$

在这恒等式中,

令 $x = -2$, 得 $1024 = -3A$, $A = -\frac{1024}{3}$,

令 $x = 1$, 得 $1 = 3B$, $B = \frac{1}{3}$.

于是,

$$\int \frac{x^{10}}{x^2+x-2} dx = \int \left[x^8 - x^7 + 3x^6 - 5x^5 + 11x^4 \right. \\
\left. - 21x^3 + 43x^2 - 85x + 171 \right. \\
\left. - \frac{1024}{3(x+2)} + \frac{1}{3(x-1)} \right] dx \\
= \frac{x^9}{9} - \frac{x^8}{8} + \frac{3x^7}{7} - \frac{5x^6}{6} + \frac{11x^5}{5} - \frac{21x^4}{4} + \frac{43x^3}{3} - \frac{85x^2}{2}$$

$$+ 171x + \frac{1}{3} \ln \left| \frac{x-1}{(x+2)^{1024}} \right| + C.$$

1869. $\int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx.$

解 $\frac{x^3 + 1}{x^3 - 5x^2 + 6x} = 1 + \frac{5x^2 - 6x + 1}{x^3 - 5x^2 + 6x}$

$$= 1 + \frac{5x^2 - 6x + 1}{x(x-2)(x-3)},$$

设 $\frac{5x^2 - 6x + 1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$, 通分后应有

$$5x^2 - 6x + 1 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2).$$

在这恒等式中,

令 $x = 0$, 得 $1 = 6A$, $A = \frac{1}{6}$,

令 $x = 2$, 得 $9 = -2B$, $B = -\frac{9}{2}$,

令 $x = 3$, 得 $28 = 3C$, $C = \frac{28}{3}$.

于是,

$$\begin{aligned} & \int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx \\ &= \int \left[1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)} \right] dx \\ &= x + \frac{1}{6} \ln|x| - \frac{9}{2} \ln|x-2| + \frac{28}{3} \ln|x-3| + C. \end{aligned}$$

$$1870. \int \frac{x^4}{x^4 + 5x^2 + 4} dx.$$

解 $\frac{x^4}{x^4 + 5x^2 + 4} = 1 + \frac{-(5x^2 + 4)}{(x^2 + 1)(x^2 + 4)}$.

设 $\frac{-(5x^2 + 4)}{(x^2 + 1)(x^2 + 4)} = \frac{A_1x + B_1}{x^2 + 1} + \frac{A_2x + B_2}{x^2 + 4}$, 通分

后应有

$$-(5x^2 + 4) \equiv (A_1x + B_1)(x^2 + 4) + (A_2x + B_2)(x^2 + 1).$$

比较等式两端 x 的同次幂的系数, 得

$$\begin{array}{l|l} x^3 & A_1 + A_2 = 0, \\ x^2 & B_1 + B_2 = -5, \\ x^1 & 4A_1 + A_2 = 0, \\ x^0 & 4B_1 + B_2 = -4. \end{array}$$

由此, $A_1 = 0$, $B_1 = \frac{1}{3}$, $A_2 = 0$, $B_2 = -\frac{16}{3}$.

于是,

$$\begin{aligned} & \int \frac{x^4}{x^4 + 5x^2 + 4} dx \\ &= \int \left[1 + \frac{1}{3(x^2 + 1)} - \frac{16}{3(x^2 + 4)} \right] dx \\ &= x + \frac{1}{3} \operatorname{arc} \operatorname{tg} x - \frac{8}{3} \operatorname{arc} \operatorname{tg} \frac{x}{2} + C. \end{aligned}$$

$$1871. \int \frac{x dx}{x^2 - 3x + 2}.$$

解 $\frac{x}{x^3-3x+2} = \frac{x}{(x-1)^2(x+2)}$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}, \text{ 通分后应有}$$

$$x \equiv A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

在这恒等式中,

$$\text{令 } x=1, \text{ 得 } 1=3B, B=\frac{1}{3};$$

$$\text{令 } x=-2, \text{ 得 } -2=9C, C=-\frac{2}{9},$$

比较 x^2 的系数, 得 $A+C=0$, 从而 $A=\frac{2}{9}$.

于是,

$$\int \frac{x dx}{x^3-3x+2} = \int \left[\frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)} \right] dx$$

$$= -\frac{1}{3(x-1)} + \frac{2}{9} \ln \left| \frac{x-1}{x+2} \right| + C.$$

1872. $\int \frac{x^2+1}{(x+1)^2(x-1)} dx.$

解 设 $\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$, 通分

后应有

$$x^2+1 \equiv A(x+1)(x-1) + B(x-1) + C(x+1)^2.$$

在这恒等式中,

$$\text{令 } x = -1, \text{ 得 } 2 = -2B, B = -1,$$

$$\text{令 } x = 1, \text{ 得 } 2 = 4C, C = \frac{1}{2},$$

比较 x^2 的系数, 得 $A + C = 1$, 从而 $A = \frac{1}{2}$.

于是,

$$\begin{aligned} & \int \frac{x^2 + 1}{(x+1)^2(x-1)} dx \\ &= \int \left[\frac{1}{2(x+1)} - \frac{1}{(x+1)^2} + \frac{1}{2(x+1)} \right] dx \\ &= \frac{1}{2} \ln|x^2 - 1| + \frac{1}{x+1} + C. \end{aligned}$$

$$1873. \int \left(\frac{x}{x^2 - 3x + 2} \right)^2 dx.$$

$$\text{解 } \left(\frac{x}{x^2 - 3x + 2} \right)^2 = \frac{x^2}{(x-1)^2(x-2)^2}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}, \text{ 通分后应有}$$

$$\begin{aligned} x^2 &= A(x-1)(x-2)^2 + B(x-2)^2 \\ &+ C(x-2)(x-1)^2 + D(x-1)^2. \end{aligned}$$

在这恒等式中,

$$\text{令 } x = 1 \text{ 得 } B = 1,$$

$$\text{令 } x = 2, \text{ 得 } D = 4,$$

比较 x^3 及 x^2 的系数, 得

$$A+C=0 \text{ 及 } -5A+B-4C+D=1,$$

由此, $A=4, C=-4$.

于是,

$$\begin{aligned} & \int \left(\frac{x}{x^2-3x+2} \right)^2 dx \\ &= \int \left[\frac{4}{x-1} + \frac{1}{(x-1)^2} - \frac{4}{x-2} + \frac{4}{(x-2)^2} \right] dx \\ &= 4 \ln|x-1| - \frac{1}{x-1} - 4 \ln|x-2| - \frac{4}{x-2} + C \\ &= 4 \ln \left| \frac{x-1}{x-2} \right| - \frac{5x-6}{x^2-3x+2} + C. \end{aligned}$$

$$1874. \int \frac{dx}{(x+1)(x+2)^2(x+3)^3}.$$

解 设 $\frac{1}{(x+1)(x+2)^2(x+3)^3}$

$$\begin{aligned} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{x+3} + \frac{E}{(x+3)^2} \\ &+ \frac{F}{(x+3)^3}, \text{ 通分后应有} \end{aligned}$$

$$\begin{aligned} 1 &\equiv A(x+2)^2(x+3)^3 + B(x+1)(x+2)(x+3)^3 \\ &+ C(x+1)(x+3)^3 + D(x+1)(x+2)^2 \\ &(x+3)^2 + E(x+1)(x+2)^2(x+3) \\ &+ F(x+1)(x+2)^2. \end{aligned}$$

在这恒等式中,

$$\text{令 } x=-1, \text{ 得 } 1=8A, \quad A=\frac{1}{8},$$

$$\text{令 } x=-2, \text{ 得 } 1=-C, \quad C=-1;$$

$$\text{令 } x = -3, \text{ 得 } 1 = -2F, \quad F = -\frac{1}{2},$$

比较 x^5, x^4 及 x^3 的系数, 得

$$\begin{array}{l} x^6 \mid A+B+D=0, \\ x^4 \mid 13A+12B+C+11D+E=0, \\ x^3 \mid 67A+56B+10C+47D+8E+F=0. \end{array}$$

$$\text{由此, } B=2, \quad D=-\frac{17}{8}, \quad E=-\frac{5}{4}.$$

于是,

$$\begin{aligned} & \int \frac{dx}{(x+1)(x+2)^2(x+3)^3} \\ &= \int \left[\frac{1}{8(x+1)} + \frac{2}{x+2} - \frac{1}{(x+2)^2} - \frac{17}{8(x+3)} \right. \\ & \quad \left. - \frac{5}{4(x+3)^2} - \frac{1}{2(x+3)^3} \right] dx \\ &= \frac{1}{8} \ln|x+1| + 2 \ln|x+2| + \frac{1}{x+2} - \frac{17}{8} \ln|x+3| \\ & \quad + \frac{5}{4(x+3)} + \frac{1}{4(x+3)^2} + C \\ &= \frac{1}{8} \ln \left| \frac{(x+1)(x+2)^{16}}{(x+3)^{17}} \right| + \frac{9x^2+50x+68}{4(x+2)(x+3)^2} + C. \end{aligned}$$

$$1875. \int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}.$$

$$\begin{aligned} \text{解} \quad & \frac{1}{x^5+x^4-2x^3-2x^2+x+1} = \frac{1}{(x-1)^2(x+1)^3} \\ &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}, \end{aligned}$$

通分后应有

$$1 \equiv A(x-1)(x+1)^3 + B(x+1)^3 + C(x-1)^2 \cdot (x+1)^2 + D(x-1)^2(x+1) + E(x-1)^2.$$

在这恒等式中,

$$\text{令 } x=1, \text{ 得 } 1=8B, \quad B=\frac{1}{8};$$

$$\text{令 } x=-1, \text{ 得 } 1=4E, \quad E=\frac{1}{4};$$

$$\text{令 } x=0, \text{ 得 } -A+B+C+D+E=1;$$

$$\text{令 } x=2, \text{ 得 } 27A+27B+9C+3D+E=1;$$

$$\text{令 } x=-2, \text{ 得 } 3A-B+9C-9D+9E=1;$$

$$\text{由此, } A=-\frac{3}{16}, \quad C=\frac{3}{16}, \quad D=\frac{1}{4}.$$

于是,

$$\begin{aligned} & \int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1} \\ &= \int \left[-\frac{3}{16(x-1)} + \frac{1}{8(x-1)^2} + \frac{3}{16(x+1)} \right. \\ & \quad \left. + \frac{1}{4(x+1)^2} + \frac{1}{4(x+1)^3} \right] dx \\ &= -\frac{3}{16} \ln|x-1| - \frac{1}{8(x-1)} + \frac{3}{16} \ln|x+1| \\ & \quad - \frac{1}{4(x+1)} - \frac{1}{8(x+1)} + C \\ &= \frac{3}{16} \ln \left| \frac{x+1}{x-1} \right| - \frac{3x^2+3x-2}{8(x-1)(x+1)^2} + C. \end{aligned}$$

$$1876. \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx.$$

解 设 $\frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}$, 通分后应有

$$x^2 + 5x + 4 \equiv (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1).$$

比较等式两端 x 的同次幂的系数, 得

$$\begin{array}{l|l} x^3 & A + C = 0, \\ x^2 & B + D = 1, \\ x^1 & 4A + C = 5, \\ x^0 & 4B + D = 4. \end{array}$$

由此, $A = \frac{5}{3}$, $B = 1$, $C = -\frac{5}{3}$, $D = 0$.

于是,

$$\begin{aligned} \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx &= \int \left(\frac{\frac{5}{3}x + 1}{x^2 + 1} + \frac{-\frac{5}{3}x}{x^2 + 4} \right) dx \\ &= \frac{5}{6} \ln \frac{x^2 + 1}{x^2 + 4} + \arctan x + C. \end{aligned}$$

本题如不直接用待定系数法将被积函数进行分解, 而使用其它技巧, 也可有更简单的方法. 事实上,

$$\begin{aligned} &\int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx \\ &= \int \frac{x^2 + 4}{(x^2 + 1)(x^2 + 4)} dx + 5 \int \frac{x dx}{(x^2 + 4)(x^2 + 1)} \end{aligned}$$

$$\begin{aligned}
&= \int \frac{dx}{x^2+1} + \frac{5}{2} \int \frac{d(x^2)}{(x^2+4)(x^2+1)} \\
&= \operatorname{arc} \operatorname{tg} x + \frac{5}{6} \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+4} \right) d(x^2) \\
&= \operatorname{arc} \operatorname{tg} x + \frac{5}{6} \ln \frac{x^2+1}{x^2+4} + C.
\end{aligned}$$

1877. $\int \frac{dx}{(x+1)(x^2+1)}$.

解 设 $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$, 通分后应有

$$1 \equiv A(x^2+1) + (Bx+C)(x+1).$$

比较等式两端 x 的同次幂的系数, 得

$$\begin{array}{l|l}
x^2 & A+B=0, \\
x^1 & B+C=0, \\
x^0 & A+C=1.
\end{array}$$

由此, $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = \frac{1}{2}$.

于是,

$$\begin{aligned}
\int \frac{dx}{(x+1)(x^2+1)} &= \int \left[\frac{1}{2(x+1)} - \frac{x-1}{2(x^2+1)} \right] dx \\
&= \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \operatorname{arc} \operatorname{tg} x + C \\
&= \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} + \frac{1}{2} \operatorname{arc} \operatorname{tg} x + C.
\end{aligned}$$

$$1878. \int \frac{dx}{(x^2 - 4x + 4)(x^2 - 4x + 5)}$$

解 由于 $\frac{1}{(x^2 - 4x + 4)(x^2 - 4x + 5)}$

$$= \frac{(x^2 - 4x + 5) - (x^2 - 4x + 4)}{(x^2 - 4x + 4)(x^2 - 4x + 5)}$$

$$= \frac{1}{(x-2)^2} - \frac{1}{x^2 - 4x + 5}$$

于是,

$$\int \frac{dx}{(x^2 - 4x + 4)(x^2 - 4x + 5)}$$

$$= \int \left[\frac{1}{(x-2)^2} - \frac{1}{x^2 - 4x + 5} \right] dx$$

$$= -\frac{1}{x-2} - \int \frac{d(x-2)}{(x-2)^2 + 1}$$

$$= -\frac{1}{x-2} - \operatorname{arc} \operatorname{tg}(x-2) + C,$$

本题若用待定系数法, 较麻烦一些, 也可获得同样的结果, 此处从略.

$$1879. \int \frac{x dx}{(x-1)^2(x^2 + 2x + 2)}$$

解 设 $\frac{x}{(x-1)^2(x^2 + 2x + 2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

+ $\frac{Cx+D}{x^2+2x+2}$; 通分后应有

$$x \equiv A(x-1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x-1)^2.$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^3 & A+C=0, \\ x^2 & A+B-2C+D=0, \\ x^1 & 2B+C-2D=1, \\ x^0 & -2A+2B+D=0. \end{array}$$

由此, $A = \frac{1}{25}$, $B = \frac{1}{5}$, $C = -\frac{1}{25}$, $D = -\frac{8}{25}$.

于是,

$$\begin{aligned} & \int \frac{x dx}{(x-1)^2(x^2+2x+2)} \\ &= \int \left[\frac{1}{25(x-1)} + \frac{1}{5(x-1)^2} - \frac{x+8}{25(x^2+2x+2)} \right] dx \\ &= \frac{1}{25} \ln|x-1| - \frac{1}{5(x-1)} - \frac{1}{50} \int \frac{2x+2}{x^2+2x+2} dx \\ & \quad - \frac{7}{25} \int \frac{d(x+1)}{(x+1)^2+1} \\ &= \frac{1}{25} \ln|x-1| - \frac{1}{5(x-1)} - \frac{1}{50} \ln(x^2+2x+2) \\ & \quad - \frac{7}{25} \operatorname{arc} \operatorname{tg}(x+1) + C \end{aligned}$$

$$= \frac{1}{50} \ln \frac{(x-1)^2}{x^2+2x+2} - \frac{1}{5(x-1)} - \frac{7}{25} \operatorname{arc} \operatorname{tg}(x+1) + C.$$

1880. $\int \frac{dx}{x(1+x)(1+x+x^2)}$

解 设 $\frac{1}{x(1+x)(1+x+x^2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$,

通分后应有

$$1 \equiv A(x+1)(1+x+x^2) + Bx(1+x+x^2) + x(x+1)(Cx+D).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l} x^3 \\ x^2 \\ x^1 \\ x^0 \end{array} \left\{ \begin{array}{l} A+B+C=0, \\ 2A+B+C+D=0, \\ 2A+B+D=0, \\ A=1. \end{array} \right.$$

由此, $A=1, B=-1, C=0, D=-1$.

于是,

$$\begin{aligned} & \int \frac{dx}{x(1+x)(1+x+x^2)} \\ &= \int \left(\frac{1}{x} - \frac{1}{1+x} - \frac{1}{1+x+x^2} \right) dx \\ &= \ln \left| \frac{x}{1+x} \right| - \frac{2}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$

本题也可以不用待定系数法。事实上，

$$\begin{aligned} \frac{1}{x(1+x)(1+x+x^2)} &= \frac{1}{(x+x^2)(1+x+x^2)} \\ &= \frac{1}{x+x^2} - \frac{1}{1+x+x^2} = \frac{1}{x} - \frac{1}{1+x} - \frac{1}{1+x+x^2}. \end{aligned}$$

1881. $\int \frac{dx}{x^3+1}$.

解 设 $\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ ，通分后应有

$$1 \equiv A(x^2-x+1) + (Bx+C)(x+1).$$

比较等式两端 x 的同次幂系数，得

$$\begin{cases} x^2 & A+B=0, \\ x^1 & -A+B+C=0, \\ x^0 & A+C=1. \end{cases}$$

由此， $A = \frac{1}{3}$ ， $B = -\frac{1}{3}$ ， $C = \frac{2}{3}$ 。

于是，

$$\begin{aligned} \int \frac{dx}{x^3+1} &= \int \left[\frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)} \right] dx \\ &= \frac{1}{3} \int \frac{dx}{x+1} - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2x-1}{\sqrt{3}} + C. \end{aligned}$$

1882. $\int \frac{x dx}{x^3 - 1}$.

解 设 $\frac{x}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$, 通分后应有

$$x \equiv A(x^2 + x + 1) + (Bx + C)(x - 1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^2 & A + B = 0, \\ x^1 & A - B + C = 1, \\ x^0 & A - C = 0. \end{array}$$

由此, $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $C = \frac{1}{3}$.

于是,

$$\begin{aligned} \int \frac{x}{x^3 - 1} dx &= \int \left[\frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)} \right] dx \\ &= \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$

1883. $\int \frac{dx}{x^4 - 1}$.

解 $\int \frac{dx}{x^4 - 1} = \frac{1}{2} \int \left(\frac{1}{x^2 - 1} - \frac{1}{x^2 + 1} \right) dx$

$$= \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \operatorname{arc} \operatorname{tg} x + C.$$

本题若用待定系数法，则较麻烦，从略。

1884. $\int \frac{dx}{x^4+1}.$

解 本题如用待定系数法来作，主要步骤如下：

设 $\frac{1}{x^4+1} = \frac{Ax+B}{x^2+x\sqrt{2}+1} + \frac{Cx+D}{x^2-x\sqrt{2}+1}$ ，则经计

算可求得 $A = \frac{\sqrt{2}}{4}$, $B = \frac{1}{2}$, $C = -\frac{\sqrt{2}}{4}$, $D = \frac{1}{2}$ 。于是，

$$\begin{aligned} \int \frac{dx}{x^4+1} &= \int \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2+x\sqrt{2}+1} dx + \int \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2-x\sqrt{2}+1} dx \\ &= \frac{\sqrt{2}}{4} \int \frac{\left(x + \frac{\sqrt{2}}{2}\right) dx}{\left(x + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} + \frac{1}{4} \int \frac{dx}{\left(x + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \\ &\quad - \frac{\sqrt{2}}{4} \int \frac{\left(x - \frac{\sqrt{2}}{2}\right) dx}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} + \frac{1}{4} \int \frac{dx}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \\ &= \frac{1}{4\sqrt{2}} [\ln(x^2+x\sqrt{2}+1) - \ln(x^2-x\sqrt{2}+1)] \\ &\quad + \frac{\sqrt{2}}{4} \left[\operatorname{arc} \operatorname{tg} \left(\frac{2x+\sqrt{2}}{\sqrt{2}} \right) \right. \\ &\quad \left. + \operatorname{arc} \operatorname{tg} \left(\frac{2x-\sqrt{2}}{\sqrt{2}} \right) \right] + C \end{aligned}$$

$$= \frac{1}{4\sqrt{2}} \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} + \frac{\sqrt{2}}{4} \operatorname{arc\,tg} \left(\frac{x\sqrt{2}}{1-x^2} \right) + C.$$

如应用下列解法，则更简单些。

$$\begin{aligned} \int \frac{dx}{x^4+1} &= \frac{1}{2} \int \frac{x^2+1}{x^4+1} dx - \frac{1}{2} \int \frac{x^2-1}{x^4+1} dx \\ &= \frac{1}{2\sqrt{2}} \operatorname{arc\,tg} \left(\frac{x^2-1}{x\sqrt{2}} \right)^{*)} \\ &\quad - \frac{1}{4\sqrt{2}} \ln \frac{x^2 - x\sqrt{2} + 1^{**)}}{x^2 + x\sqrt{2} + 1} + C_1, \end{aligned}$$

注意到 $\operatorname{arc\,tg} \left(\frac{x^2-1}{x\sqrt{2}} \right) = \frac{\pi}{2} + \operatorname{arc\,tg} \left(\frac{x\sqrt{2}}{1-x^2} \right)$ ，最后

即得

$$\begin{aligned} \int \frac{dx}{x^4+1} &= \frac{1}{2\sqrt{2}} \operatorname{arc\,tg} \left(\frac{x\sqrt{2}}{1-x^2} \right) \\ &\quad + \frac{1}{4\sqrt{2}} \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} + C. \end{aligned}$$

*) 利用1712题的结果。

**) 利用1713题的结果。

1885. $\int \frac{dx}{x^4 + x^2 + 1}$.

解 设 $\frac{1}{x^4 + x^2 + 1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$ ，通分后

应有

$$1 \equiv (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^3 & A+C=0, \\ x^2 & -A+B+C+D=0, \\ x^1 & A-B+C+D=0, \\ x^0 & B+D=1. \end{array}$$

由此, $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$, $D = \frac{1}{2}$.

于是,

$$\begin{aligned} \int \frac{dx}{x^4+x^2+1} &= \int \frac{\frac{1}{2}(x+1)}{x^2+x+1} dx - \int \frac{\frac{1}{2}(x-1)}{x^2-x+1} dx \\ &= \frac{1}{4} \int \frac{(2x+1) dx}{x^2+x+1} + \frac{1}{4} \int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &\quad - \frac{1}{4} \int \frac{(2x-1) dx}{x^2-x+1} + \frac{1}{4} \int \frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{4} \left[\ln(x^2+x+1) - \ln(x^2-x+1) \right] \\ &\quad + \frac{1}{2\sqrt{3}} \left[\operatorname{arc\,tg}\left(\frac{2x+1}{\sqrt{3}}\right) + \operatorname{arc\,tg}\left(\frac{2x-1}{\sqrt{3}}\right) \right] + C_1 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + \frac{1}{2\sqrt{3}} \operatorname{arc\,tg} \left(\frac{\sqrt{3}x}{1-x^2} \right) + C_1 \\
&= \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + \frac{1}{2\sqrt{3}} \operatorname{arc\,tg} \left(\frac{x^2 - 1}{x\sqrt{3}} \right) + C.
\end{aligned}$$

如不用待定系数法解本题，则更简单些，解法与上题类似：

$$\begin{aligned}
\int \frac{dx}{x^4 + x^2 + 1} &= \frac{1}{2} \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx \\
&\quad - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx \\
&= \frac{1}{2} \int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \\
&= \frac{1}{2} \int \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + 3} - \frac{1}{2} \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 1} \\
&= \frac{1}{2\sqrt{3}} \operatorname{arc\,tg} \left(\frac{x^2 - 1}{x\sqrt{3}} \right) + \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + C.
\end{aligned}$$

1886. $\int \frac{dx}{x^6 + 1}$

解 本题如用待定系数法来作，运算较麻烦，经计算可得

$$\frac{1}{x^6 + 1} = \frac{1}{3(x^2 + 1)} + \frac{\frac{\sqrt{3}}{6}x + \frac{1}{3}}{x^2 + x\sqrt{3} + 1}$$

$$-\frac{\sqrt{3}}{6}x + \frac{1}{3} + \frac{1}{x^2 - x\sqrt{3} + 1},$$

积分步骤与1884题与1885题完全类似，不再详解，其

结果为 $\frac{1}{2} \operatorname{arc} \operatorname{tg} x + \frac{1}{6} \operatorname{arc} \operatorname{tg}(x^3)$

$$+ \frac{1}{4\sqrt{3}} \ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + C.$$

本题如不用待定系数法来作，则更简单些。下面列举两种解法：

解法一

$$\begin{aligned} \int \frac{dx}{x^6 + 1} &= \frac{1}{2} \int \frac{x^4 + 1}{x^6 + 1} dx - \frac{1}{2} \int \frac{x^4 - 1}{x^6 + 1} dx \\ &= \frac{1}{2} \int \frac{x^2 + (x^4 - x^2 + 1)}{x^6 + 1} dx \\ &= \frac{1}{2} \int \frac{(x^2 - 1)(x^2 + 1)}{(x^2 + 1)(x^4 - x^2 + 1)} dx \\ &= \frac{1}{2} \int \frac{x^2}{x^6 + 1} dx + \frac{1}{2} \int \frac{dx}{1 + x^2} - \frac{1}{2} \int \frac{x^2 - 1}{x^4 - x^2 + 1} dx \\ &= \frac{1}{6} \int \frac{d(x^3)}{1 + (x^3)^2} + \frac{1}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 - 3} \\ &= \frac{1}{6} \operatorname{arc} \operatorname{tg}(x^3) + \frac{1}{2} \operatorname{arc} \operatorname{tg} x \end{aligned}$$

$$+ \frac{1}{4\sqrt{3}} \ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + C.$$

解法二

仿照1881题的分解法，可得

$$\frac{1}{x^6 + 1} = \frac{1}{3(x^2 + 1)} - \frac{x^2 - 2}{3(x^4 - x^2 + 1)}.$$

于是，

$$\begin{aligned} \int \frac{dx}{x^6 + 1} &= \frac{1}{3} \int \frac{dx}{x^2 + 1} - \frac{1}{3} \int \frac{(x^2 - 2)dx}{x^4 - x^2 + 1} \\ &= \frac{1}{3} \operatorname{arc} \operatorname{tg} x - \frac{1}{6} \int \frac{(x^2 + 1) + (x^2 - 1)}{x^4 - x^2 + 1} dx \\ &\quad + \frac{1}{3} \int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 - x^2 + 1} dx \\ &= \frac{1}{3} \operatorname{arc} \operatorname{tg} x + \frac{1}{6} \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx \\ &\quad - \frac{1}{2} \int \frac{x^2 - 1}{x^4 - x^2 + 1} dx \\ &= \frac{1}{3} \operatorname{arc} \operatorname{tg} x + \frac{1}{6} \int \frac{d(x - \frac{1}{x})}{(x - \frac{1}{x})^2 + 1} - \frac{1}{2} \int \frac{d(x + \frac{1}{x})}{(x + \frac{1}{x})^2 - 3} \\ &= \frac{1}{3} \operatorname{arc} \operatorname{tg} x + \frac{1}{6} \operatorname{arc} \operatorname{tg} \left(\frac{x^2 - 1}{x} \right) \\ &\quad + \frac{1}{4\sqrt{3}} \ln \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + C. \end{aligned}$$

两种答案形式不同，实质上是一致的。

$$1887. \int \frac{dx}{(1+x)(1+x^2)(1+x^3)}.$$

$$\text{解 设 } \frac{1}{(1+x)(1+x^2)(1+x^3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \\ + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2-x+1},$$

通分后应有

$$1 \equiv A(x+1)(x^2+1)(x^2-x+1) + B(x^2+1) \\ (x^2-x+1) + (Cx+D)(x+1)^2(x^2-x+1) \\ + (Ex+F)(x+1)^2(x^2+1).$$

比较等式两端 x 的同次幂系数，得

$$\begin{array}{l} x^5 \quad A+C+E=0, \\ x^4 \quad B+C+D+2E+F=0, \\ x^3 \quad A+D+2E+2F-B=0, \\ x^2 \quad A+2B+C+2E+2F=0, \\ x^1 \quad -B+C+D+E+2F=0, \\ x^0 \quad A+B+D+F=1. \end{array}$$

由此， $A=\frac{1}{3}$ ， $B=\frac{1}{6}$ ， $C=0$ ， $D=\frac{1}{2}$ ， $E=-\frac{1}{3}$ ， $F=0$ 。

于是，

$$\int \frac{dx}{(1+x)(1+x^2)(1+x^3)} = \int \left[\frac{1}{3(x+1)} + \frac{1}{6(x+1)^2} \right]$$

$$\begin{aligned}
& + \frac{1}{2(x^2+1)} - \frac{x}{3(x^2-x+1)} \Big] dx \\
& = \frac{1}{3} \ln|1+x| - \frac{1}{6(x+1)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} x \\
& \quad - \frac{1}{6} \int \frac{(2x-1)dx}{x^2-x+1} - \frac{1}{6} \int \frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\
& = \frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} - \frac{1}{6(x+1)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} x \\
& \quad - \frac{1}{3\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2x-1}{\sqrt{3}} \right) + C.
\end{aligned}$$

1888. $\int \frac{dx}{x^6-x^4+x^3-x^2+x-1}$

解 设 $\frac{1}{x^6-x^4+x^3-x^2+x-1}$

$$= \frac{1}{(x-1)(x^2-x+1)(x^2+x+1)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{Cx+D}{x^2-x+1},$$

通分后应有

$$\begin{aligned}
1 & \equiv A(x^2+x+1)(x^2-x+1) + (Bx+C)(x-1) \\
& \quad (x^2-x+1) + (Dx+E)(x-1)(x^2+x+1).
\end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^4 & A+B+D=0, \\ x^3 & -2B+C+E=0, \\ x^2 & A+2B-2C=0, \\ x^1 & -B+2C-D=0, \\ x^0 & A-C-E=1. \end{array}$$

由此, $A=\frac{1}{3}, B=-\frac{1}{3}, C=-\frac{1}{6}, D=0, E=-\frac{1}{2}$.

于是,

$$\begin{aligned} & \int \frac{dx}{x^5 - x^4 + x^3 - x^2 + x - 1} \\ &= \int \left[\frac{1}{3(x-1)} - \frac{2x+1}{6(x^2+x+1)} - \frac{1}{2(x^2+x+1)} \right] dx \\ &= \frac{1}{6} \ln \frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2x-1}{\sqrt{3}} \right) + C. \end{aligned}$$

1889. $\int \frac{x^2 dx}{x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1}$

解 设 $\frac{x^2}{x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1} = \frac{Ax+B}{x^2+2x+2}$

+ $\frac{Cx+D}{x^2+x+\frac{1}{2}}$, 通分后应有

$$x^2 = (Ax+B)(x^2+x+\frac{1}{2})$$

$$+ (Cx + D)(x^2 + 2x + 2).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{cases} x^3 & A + C = 0, \\ x^2 & A + B + 2C + D = 1, \\ x^1 & \frac{A}{2} + B + 2C + 2D = 0, \\ x^0 & \frac{B}{2} + 2D = 0. \end{cases}$$

由此, $A = \frac{4}{5}, B = \frac{12}{5}, C = -\frac{4}{5}, D = -\frac{3}{5}.$

于是,

$$\begin{aligned} & \int \frac{x^2 dx}{x^4 + 3x^3 + \frac{9}{2}x^2 + 3x + 1} \\ &= \int \left[\frac{4(x+3)}{5(x^2 + 2x + 2)} - \frac{4x+3}{5(x^2 + x + \frac{1}{2})} \right] dx \\ &= \frac{2}{5} \int \frac{(2x+2) dx}{x^2 + 2x + 2} + \frac{8}{5} \int \frac{d(x+1)}{(x+1)^2 + 1} \\ & \quad - \frac{2}{5} \int \frac{(2x+1) dx}{x^2 + x + \frac{1}{2}} - \frac{1}{5} \int \frac{d(x + \frac{1}{2})}{(x + \frac{1}{2})^2 + \frac{1}{4}} \\ &= \frac{2}{5} \ln \frac{x^2 + 2x + 2}{x^2 + x + \frac{1}{2}} + \frac{8}{5} \operatorname{arc} \operatorname{tg}(x+1) \end{aligned}$$

$$-\frac{2}{5} \operatorname{arc} \operatorname{tg}(2x+1) + C.$$

1890. 在甚么条件下, 积分

$$\int \frac{ax^2 + bx + c}{x^3(x-1)^2} dx$$

为有理函数?

解 设 $\frac{ax^2 + bx + c}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2},$

通分后应有

$$ax^2 + bx + c \equiv Ax^2(x-1)^2 + Bx(x-1)^2 + C(x-1)^2 + Dx^3(x-1) + Ex^3,$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l} x^4 | A + D = 0, \\ x^3 | -2A + B - D + E = 0, \\ x^2 | A - 2B + C = a, \\ x^1 | B - 2C = b, \\ x^0 | C = c. \end{array}$$

由此, $A = a + 2b + 3c, B = b + 2c, C = c,$

$$D = -(a + 2b + 3c), E = a + b + c.$$

当 $A = D = 0$, 即 $a + 2b + 3c = 0$ 时, 积分

$$\int \frac{ax^2 + bx + c}{x^3(x-1)^2} dx$$

为有理函数.

利用奥斯特洛格拉得斯基方法*，计算积分：

$$1891. \int \frac{x dx}{(x-1)^2(x+1)^3}.$$

解 $Q = (x-1)^2(x+1)^3,$

$$Q_1 = (x-1)(x+1)^2 = x^3 + x^2 - x - 1,$$

$$Q_2 = (x-1)(x+1) = x^2 - 1.$$

设 $\frac{x}{(x-1)^2(x+1)^3}$

$$= \left(\frac{Ax^2 + Bx + C}{x^3 + x^2 - x - 1} \right)' + \frac{Dx + E}{x^2 - 1}, \text{ 从而}$$

$$x \equiv (2Ax + B)(x-1)(x+1) - (3x-1)$$

$$\cdot (Ax^2 + Bx + C) + (Dx + E)(x-1)(x+1)^2.$$

比较等式两端 x 的同次幂系数，得

* 所谓奥氏方法，是指关于有理真分式 $\frac{P(x)}{Q(x)}$ 的积分，可以借助代数方法来分离成一个真分式与另一个真分式积分的和，使得在新的被积真分式函数中，其分母次数达到最低状态，也即在公式

$$\int \frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)} dx \quad (1)$$

中，如果 $P(x), Q(x)$ 已知，且设分母 $Q(x)$ 可以分解成一次与二次类型的实因式：

$$Q(x) = (x-a)^k \cdots (x^2+px+q)^m \cdots.$$

其中 k, \dots, m, \dots 是自然数。在公式 (1) 的右端分母已知，形如：

$$Q_1(x) = (x-a)^{k-1} \cdots (x^2+px+q)^{m-1} \cdots,$$

$$Q_2(x) = (x-a) \cdots (x^2+px+q) \cdots,$$

且满足 $Q_1(x) \cdot Q_2(x) = Q(x)$ 。而 $P_1(x)$ 和 $P_2(x)$ 为相应比 $Q_1(x)$ 和 $Q_2(x)$ 更低次的多项式，一般可用待定系数法求得。这种利用公式 (1) 来求积分的方法，就是所谓的奥斯特洛格拉得斯基方法。详细可以参见 $\Gamma. M.$ 菲赫金哥尔茨著（北大译），微积分学教程，第二卷一分册，第264目。——题解编者注

$$\begin{array}{l}
 x^4 \quad D = 0, \\
 x^3 \quad -A + D + E = 0, \\
 x^2 \quad A - 2B - D + E = 0, \\
 x^1 \quad -2A - 3C + B - D - E = 1, \\
 x^0 \quad -B - C - E = 0.
 \end{array}$$

由此, $A = -\frac{1}{8}$, $B = -\frac{1}{8}$, $C = -\frac{1}{4}$, $D = 0$, $E = -\frac{1}{8}$.

于是,

$$\begin{aligned}
 \int \frac{x dx}{(x-1)^2 (x+1)^3} &= -\frac{x^2 + x + 2}{8(x-1)(x+1)^2} \\
 &- \frac{1}{8} \int \frac{dx}{x^2 - 1} \\
 &= -\frac{x^2 + x + 2}{8(x-1)(x+1)^2} + \frac{1}{16} \ln \left| \frac{x+1}{x-1} \right| + C.
 \end{aligned}$$

1892. $\int \frac{dx}{(x^3+1)^2}$

解 $Q = (x+1)^2(x^2-x+1)^2$,

$$Q_1 = Q_2 = x^3 + 1.$$

设 $\frac{1}{(x^3+1)^2} = \left(\frac{Ax^2+Bx+C}{x^3+1} \right)' + \frac{Dx^2+Ex+F}{x^3+1}$, 从而

$$\begin{aligned}
 1 &\equiv (2Ax+B)(x^3+1) - 3x^2(Ax^2+Bx+C) \\
 &+ (Dx^2+Ex+F)(x^3+1).
 \end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l}
 x^5 \quad D = 0, \\
 x^4 \quad -A + E = 0, \\
 x^3 \quad -2B + F = 0, \\
 x^2 \quad -3C + D = 0, \\
 x^1 \quad 2A + E = 0, \\
 x^0 \quad B + F = 1.
 \end{array}$$

由此, $A=0, B=\frac{1}{3}, C=0, D=0, E=0, F=\frac{2}{3}$.

于是,

$$\begin{aligned}
 \int \frac{dx}{(x^3+1)^2} &= \frac{x}{3(x^3+1)} + \frac{2}{3} \int \frac{dx}{x^3+1} \\
 &= \frac{x}{3(x^3+1)} + \frac{1}{9} \ln \frac{(x+1)^2}{x^2-x+1} \\
 &\quad + \frac{2}{3\sqrt{3}} \operatorname{arc\,tg} \left(\frac{2x-1}{\sqrt{3}} \right)^{*} + C.
 \end{aligned}$$

*) 利用1881题的结果。

1893. $\int \frac{dx}{(x^2+1)^3}$.

解 $Q=(x^2+1)^3, Q_1=(x^2+1)^2, Q_2=x^2+1$.

设 $\frac{1}{(x^2+1)^3} = \left[\frac{Ax^3+Bx^2+Cx+D}{(x^2+1)^2} \right]' + \frac{Ex+F}{x^2+1}$,

从而

$$1 \equiv (3Ax^2+2Bx+C)(x^2+1) - 4x(Ax^3+Bx^2)$$

$$+Cx+D)+(Ex+F)(x^2+1)^2.$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l} x^5 \quad F=0, \\ x^4 \quad -A+F=0, \\ x^3 \quad -2B+2E=0, \\ x^2 \quad 3A-3C+2F=0, \\ x^1 \quad 2B-4D+E=0, \\ x^0 \quad C+F=1. \end{array}$$

由此, $A=\frac{3}{8}, B=0, C=\frac{5}{8}, D=0, E=0, F=\frac{3}{8}$.

$$\begin{aligned} \int \frac{dx}{(x^2+1)^3} &= \frac{x(3x^2+5)}{8(x^2+1)^3} + \frac{3}{8} \int \frac{dx}{x^2+1} \\ &= \frac{x(3x^2+5)}{8(x^2+1)^3} + \frac{3}{8} \operatorname{arctg} x + C. \end{aligned}$$

1894. $\int \frac{x^2 dx}{(x^2+2x+2)^2}$.

解 $Q=(x^2+2x+2)^2, Q_1=Q_2=x^2+2x+2$.

设 $\frac{x^2}{(x^2+2x+2)^2} = \left(\frac{Ax+B}{x^2+2x+2} \right)' + \frac{Cx+D}{x^2+2x+2}$,

从而

$$\begin{aligned} x^2 &\equiv A(x^2+2x+2) - 2(x+1)(Ax+B) \\ &\quad + (Cx+D)(x^2+2x+2). \end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^3 & C=0, \\ x^2 & -A+2C+D=1, \\ x^1 & -2B+2C+2D=0, \\ x^0 & 2A-2B+2D=0. \end{array}$$

由此, $A=0$, $B=1$, $C=0$, $D=1$.

于是,

$$\begin{aligned} \int \frac{x^2 dx}{(x^2+2x+2)^2} &= \frac{1}{x^2+2x+2} + \int \frac{dx}{x^2+2x+2} \\ &= \frac{1}{x^2+2x+2} + \int \frac{d(x+1)}{(x+1)^2+1} \\ &= \frac{1}{x^2+2x+2} + \operatorname{arc} \operatorname{tg}(x+1) + C. \end{aligned}$$

本题如不用奥斯特洛格拉得斯基方法, 则更容易得出上述结果. 事实上,

$$\begin{aligned} \int \frac{x^2 dx}{(x^2+2x+2)^2} &= \int \frac{(x^2+2x+2) - (2x+2)}{(x^2+2x+2)^2} dx \\ &= \int \frac{dx}{x^2+2x+2} - \int \frac{(2x+2)dx}{(x^2+2x+2)^2} \\ &= \int \frac{d(x+1)}{(x+1)^2+1} - \int \frac{d(x^2+2x+2)}{(x^2+2x+2)^2} \\ &= \operatorname{arc} \operatorname{tg}(x+1) + \frac{1}{x^2+2x+2} + C. \end{aligned}$$

1895. $\int \frac{dx}{(x^4+1)^2}$

解 $Q = (x^4 + 1)^2$, $Q_1 = Q_2 = x^4 + 1$,

设 $\frac{1}{(x^4 + 1)^2} = \left(\frac{Ax^3 + Bx^2 + Cx + D}{x^4 + 1} \right)'$
 $+ \frac{Ex^3 + Fx^2 + Gx + H}{x^4 + 1}$, 从而

$$1 \equiv (3Ax^2 + 2Bx + C)(x^4 + 1) - 4x^3(Ax^3 + Bx^2 + Cx + D) + (Ex^3 + Fx^2 + Gx + H)(x^4 + 1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{array}{l|l} x^7 & E = 0, \\ x^6 & -A + F = 0, \\ x^5 & -2B + G = 0, \\ x^4 & -3C + H = 0, \\ x^3 & -4D + E = 0, \\ x^2 & 3A + F = 0, \\ x^1 & 2B + G = 0, \\ x^0 & C + H = 1. \end{array}$$

由此, $A = 0$, $B = 0$, $C = \frac{1}{4}$, $D = 0$, $E = 0$, $F = 0$,

$$G = 0, \quad H = \frac{3}{4}.$$

于是,

$$\begin{aligned} \int \frac{dx}{(x^4 + 1)^2} &= \frac{x}{4(x^4 + 1)} + \frac{3}{4} \int \frac{dx}{x^4 + 1} \\ &= \frac{x}{4(x^4 + 1)} + \frac{3}{16\sqrt{2}} \ln \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} \end{aligned}$$

$$-\frac{3}{8\sqrt{2}} \arctg \frac{x\sqrt{2}}{x^2-1} + C$$

*) 利用1884题的结果.

$$1896. \int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2} dx.$$

解 $Q=(x-1)(x^2+x+1)^2, Q_1=x^2+x+1,$
 $Q_2=(x-1)(x^2+x+1).$

设 $\frac{x^2+3x-2}{(x-1)(x^2+x+1)^2} = \left(\frac{Ax+B}{x^2+x+1} \right)'$
 $+ \frac{Cx^2+Dx+E}{(x-1)(x^2+x+1)},$ 从而

$$x^2+3x-2 \equiv A(x-1)(x^2+x+1) - (2x+1) \cdot (Ax+B)(x-1) + (Cx^2+Dx+E)(x^2+x+1).$$

比较等式两端 x 的同次幂系数, 得

$$\begin{aligned} x^4 & C=0, \\ x^3 & -A+C+D=0, \\ x^2 & A-2B+C+D+E=1, \\ x^1 & A+B+D+E=3, \\ x^0 & -A+B+E=-2. \end{aligned}$$

由此, $A=\frac{5}{3}, B=\frac{2}{3}, C=0, D=\frac{5}{3}, E=-1.$

再将 $\frac{\frac{5}{3}x-1}{(x-1)(x^2+x+1)}$ 分解, 可得

$$\frac{\frac{5}{3}x-1}{(x-1)(x^2+x+1)} = \frac{2}{9(x-1)} - \frac{2x-11}{9(x^2+x+1)}$$

于是,

$$\begin{aligned} & \int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2} dx \\ &= \frac{5x+2}{3(x^2+x+1)} + \frac{2}{9} \int \frac{dx}{x-1} - \frac{1}{9} \int \frac{2x-11}{x^2+x+1} dx \\ &= \frac{5x+2}{3(x^2+x+1)} + \frac{2}{9} \ln|x-1| - \frac{1}{9} \int \frac{2x+1}{x^2+x+1} dx \\ & \quad + \frac{4}{3} \int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{5x+2}{3(x^2+x+1)} + \frac{1}{9} \ln \frac{(x-1)^2}{x^2+x+1} \\ & \quad + \frac{8}{3\sqrt{3}} \arctg\left(\frac{2x+1}{\sqrt{3}}\right) + C. \end{aligned}$$

1897. $\int \frac{dx}{(x^4-1)^3}$.

解 $Q = (x^4-1)^3, Q_1 = (x^4-1)^2, Q_2 = x^4-1$.

设 $\frac{1}{(x^4-1)^3} = \left[\frac{P(x)}{(x^4-1)^2} \right]' + \frac{P_1(x)}{x^4-1}$, 其中

$$P(x) = Ax^7 + Bx^6 + Cx^5 + Dx^4 + Ex^3$$

$$+ Fx^2 + Gx + H,$$

$$P_1(x) = A_1x^3 + B_1x^2 + C_1x + D_1,$$

从而利用待定系数法，解出 $A=0$ ， $B=0$ ， C
 $=\frac{7}{32}$ ， $D=0$ ， $E=0$ ， $F=0$ ， $G=-\frac{11}{32}$ ， $H=0$ ，
 $A_1=0$ ， $B_1=0$ ， $C_1=0$ ， $D_1=\frac{21}{32}$ 。

于是，

$$\begin{aligned} \int \frac{dx}{(x^4-1)^3} &= \frac{7x^5-11x}{32(x^4-1)^2} + \frac{21}{32} \int \frac{dx}{x^4-1} \\ &= \frac{7x^5-11x}{32(x^4-1)^2} + \frac{21}{128} \ln \left| \frac{x-1}{x+1} \right| - \frac{21}{64} \operatorname{arc} \operatorname{tg} x^{**}) + C. \end{aligned}$$

*) 利用1883题的结果。

分出下列积分的代数部分：

$$1898. \int \frac{x^2+1}{(x^4+x^2+1)^2} dx.$$

$$\begin{aligned} \text{解 设 } \int \frac{x^2+1}{(x^4+x^2+1)^2} dx &= \frac{Ax^3+Bx^2+Cx+D}{x^4+x^2+1} \\ &+ \int \frac{A_1x^3+B_1x^2+C_1x+D_1}{x^4+x^2+1} dx. \end{aligned}$$

上述等式右端的积分为非代数部分，因此，只需要求出 A, B, C, D 就可以了。等式两端求导并通分，得

$$\begin{aligned} x^2+1 &\equiv (3Ax^2+2Bx+C)(x^4+x^2+1) \\ &- (4x^3+2x)(Ax^3+Bx^2+Cx+D) \\ &+ (A_1x^3+B_1x^2+C_1x+D_1)(x^4+x^2+1). \end{aligned}$$

比较等式两端 x 的同次幂系数, 解出 $A = \frac{1}{6}$, $B = 0$,

$C = \frac{1}{3}$, $D = 0$, $A_1 = 0$, $B_1 = -\frac{1}{6}$, $C_1 = 0$, $D_1 = \frac{2}{3}$. 因

此, 所求积分的代数部分为

$$\frac{x^3 + 2x}{6(x^4 + x^2 + 1)}.$$

1899⁺. $\int \frac{dx}{(x^3 + x + 1)^3}.$

解 设 $\int \frac{dx}{(x^3 + x + 1)^3}$

$$= \frac{Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F}{(x^3 + x + 1)^2}$$

$$+ \int \frac{Gx^2 + Hx + L}{x^3 + x + 1} dx.$$

对上述等式两端求导再通分, 得

$$\begin{aligned} 1 &\equiv (5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E)(x^3 + x + 1) \\ &\quad - 2(3x^2 + 1)(Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F) \\ &\quad + (Gx^2 + Hx + L)(x^3 + x + 1)^2. \end{aligned}$$

比较等式两端 x 的同次幂系数, 解出 $A = -\frac{243}{961}$,

$$B = \frac{357}{1922}, \quad C = -\frac{405}{961}, \quad D = -\frac{315}{1922}, \quad E = \frac{156}{961}, \quad F$$

$= -\frac{224}{961}, \quad G = 0, \quad H = -\frac{243}{961}, \quad L = \frac{357}{961}$. 因此, 所

求积分的代数部分为

$$\frac{486x^5 + 357x^4 + 810x^3 + 315x^2 - 312x + 448}{1922(x^5 + x + 1)^2}$$

1900. $\int \frac{4x^5 - 1}{(x^5 + x + 1)^2} dx.$

解 设 $\int \frac{4x^5 - 1}{(x^5 + x + 1)^2} dx$

$$= \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{x^5 + x + 1}$$

$$+ \int \frac{Fx^4 + Gx^3 + Hx^2 + Lx + M}{x^5 + x + 1} dx.$$

对上述等式两端求导再通分, 得

$$\begin{aligned} 4x^5 - 1 &\equiv (4Ax^3 + 3Bx^2 + 2Cx + D)(x^5 + x + 1) \\ &- (5x^4 + 1)(Ax^4 + Bx^3 + Cx^2 + Dx + E) \\ &+ (Fx^4 + Gx^3 + Hx^2 + Lx + M)(x^5 + x + 1). \end{aligned}$$

比较等式两端 x 的同次幂系数, 解出 $A=0$, $B=0$, $C=0$, $D=-1$, $E=0$, $F=0$, $G=0$, $H=0$, $L=0$, $M=0$. 因此, 所求积分的代数部分为

$$-\frac{x}{x^5 + x + 1} \quad (\text{全部积分}).$$

1901. 计算积分

$$\int \frac{dx}{x^4 + 2x^3 + 3x^2 + 2x + 1}.$$

解 $Q = x^4 + 2x^3 + 3x^2 + 2x + 1 = (x^2 + x + 1)^2,$

$$Q_1 = Q_2 = x^2 + x + 1.$$

$$\text{设 } \frac{1}{x^4 + 2x^3 + 3x^2 + 2x + 1} \\ = \left(\frac{Ax + B}{x^2 + x + 1} \right)' + \frac{Cx + D}{x^2 + x + 1}, \text{ 从而}$$

$$1 \equiv A(x^2 + x + 1) - (2x + 1)(Ax + B) \\ + (Cx + D)(x^2 + x + 1).$$

比较等式两端 x 的同次幂系数, 解出 $A = \frac{2}{3}$, $B = \frac{1}{3}$,

$C = 0$, $D = \frac{2}{3}$. 于是,

$$\int \frac{dx}{x^4 + 2x^3 + 3x^2 + 2x + 1} \\ = \frac{2x + 1}{3(x^2 + x + 1)} + \frac{2}{3} \int \frac{dx}{x^2 + x + 1} \\ = \frac{2x + 1}{3(x^2 + x + 1)} + \frac{2}{3} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\ = \frac{2x + 1}{3(x^2 + x + 1)} + \frac{4}{3\sqrt{3}} \operatorname{arctg}\left(\frac{2x + 1}{\sqrt{3}}\right) + C.$$

1902⁺. 在甚么条件下, 积分

$$\int \frac{ax^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2} dx$$

为有理函数?

解 (1) 当 $a \neq 0$ 且 $b^2 - ac = 0$ 时, $ax^2 + 2bx + c$

$=a(x-x_0)^2$, 其中 x_0 为实数. 此时

$$\frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2}$$

$$= \frac{\alpha(x-x_0)^2 + 2\alpha x_0(x-x_0) + \alpha x_0^2 + 2\beta(x-x_0) + 2\beta x_0 + \gamma}{a^2(x-x_0)^4}$$

$$= \frac{\alpha}{a^2(x-x_0)^2} + \frac{2\alpha x_0 + 2\beta}{a^2(x-x_0)^3} + \frac{\alpha x_0^2 + 2\beta x_0 + \gamma}{a^2(x-x_0)^4},$$

从而积分为有理函数.

(2) 当 $a \neq 0$ 且 $b^2 - ac \neq 0$ 时, 则设

$$\frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2}$$

$$= \left(\frac{Ax + B}{ax^2 + 2bx + c} \right)' + \frac{Cx + D}{ax^2 + 2bx + c},$$

从而

$$\alpha x^2 + 2\beta x + \gamma \equiv A(ax^2 + 2bx + c) - (2ax + 2b)$$

$$(Ax + B) + (Cx + D)(ax^2 + 2bx + c),$$

比较等式两端 x 的同次幂系数, 可解得 $C = 0$,

$$D = \frac{2b\beta - a\gamma - ca}{2(b^2 - ac)}. \text{ 从而当 } a\gamma + ca = 2b\beta \text{ 时 } D = 0,$$

此时积分为有理函数.

(3) 当 $a = 0$, $b \neq 0$ 时,

$$\frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2 + 2bx + c)^2}$$

$$\begin{aligned}
&= \frac{\alpha\left(x + \frac{c}{2b}\right)^2 - \frac{\alpha c}{b}\left(x + \frac{c}{2b}\right) + \frac{\alpha c^2}{4b^2} + 2\beta\left(x + \frac{c}{2b}\right) - \frac{\beta c}{b} + \gamma}{4b^2\left(x + \frac{c}{2b}\right)^2} \\
&= \frac{\alpha}{4b^2} + \frac{2\beta - \frac{\alpha c}{b}}{4b^2\left(x + \frac{c}{2b}\right)} + \frac{\frac{\alpha c^2}{4b^2} - \frac{\beta c}{b} + \gamma}{4b^2\left(x + \frac{c}{2b}\right)^2}.
\end{aligned}$$

故当 $2\beta - \frac{\alpha c}{b} = 0$ 即 $\alpha c = 2b\beta$ 时, 积分为有理函数. 这

种情况可归并到 $\alpha\gamma + c\alpha = 2b\beta$ 中去.

(4) 当 $a = b = 0, c \neq 0$ 时, 积分显然为有理函数, 这种情况可归并到 $b^2 - ac = 0$ 中去.

综上所述, 当 $b^2 - ac = 0$ 或 $\alpha\gamma + c\alpha = 2b\beta$ 时, 积分为有理函数.

利用各种方法, 计算下列积分:

1903. $\int \frac{x^3}{(x-1)^{100}} dx.$

$$\begin{aligned}
\text{解} \quad \int \frac{x^3}{(x-1)^{100}} dx &= \int \frac{[(x-1)+1]^3}{(x-1)^{100}} dx \\
&= \int \left[\frac{1}{(x-1)^{97}} + \frac{3}{(x-1)^{98}} + \frac{3}{(x-1)^{99}} \right. \\
&\quad \left. + \frac{1}{(x-1)^{100}} \right] dx \\
&= -\frac{1}{96(x-1)^{96}} - \frac{3}{97(x-1)^{97}} - \frac{3}{98(x-1)^{98}}
\end{aligned}$$

$$-\frac{1}{99(x-1)^{99}} + C.$$

1904. $\int \frac{x dx}{x^8 - 1}.$

解 $\int \frac{x dx}{x^8 - 1} = \frac{1}{2} \int \frac{d(x^2)}{(x^2)^4 - 1}$

$$= \frac{1}{8} \ln \left| \frac{x^2 - 1}{x^2 + 1} \right| - \frac{1}{4} \operatorname{arctg}(x^2) + C.$$

*) 利用1883题的结果。

1905. $\int \frac{x^3 dx}{x^8 + 3}.$

解 $\int \frac{x^3 dx}{x^8 + 3} = \frac{1}{4} \int \frac{d(x^4)}{(x^4)^2 + 3}$

$$= \frac{1}{4\sqrt{3}} \operatorname{arctg} \left(\frac{x^4}{\sqrt{3}} \right) + C.$$

1906. $\int \frac{x^2 + x}{x^6 + 1} dx.$

解 $\int \frac{x^2 + x}{x^6 + 1} dx = \frac{1}{3} \int \frac{d(x^3)}{(x^3)^2 + 1} + \frac{1}{2} \int \frac{d(x^2)}{(x^2)^3 + 1}$

$$= \frac{1}{3} \operatorname{arctg}(x^3) + \frac{1}{2} \left[\frac{1}{6} \ln \frac{(x^2 + 1)^2}{x^4 - x^2 + 1} \right.$$

$$\left. + \frac{1}{\sqrt{3}} \operatorname{arctg} \left(\frac{2x^2 - 1}{\sqrt{3}} \right) \right] + C$$

$$= \frac{1}{3} \operatorname{arctg}(x^3) + \frac{1}{12} \ln \frac{(x^2+1)^2}{x^4-x^2+1} \\ + \frac{1}{2\sqrt{3}} \operatorname{arctg}\left(\frac{2x^2-1}{\sqrt{3}}\right) + C.$$

*) 利用1881题的结果.

1907. $\int \frac{x^4-3}{x(x^8+3x^4+2)} dx.$

解 $\int \frac{x^4-3}{x(x^8+3x^4+2)} dx = \int \frac{(1-\frac{3}{x^4})dx}{x^8(1+\frac{3}{x^4}+\frac{2}{x^8})}$

$$= \int \frac{-\frac{1}{4}(1-\frac{3}{x^4})d(\frac{1}{x^4})}{\frac{2}{x^8}+\frac{3}{x^4}+1}$$

$$= -\frac{1}{4} \int \left(\frac{5}{\frac{2}{x^4}+1} - \frac{4}{\frac{1}{x^4}+1} \right) d\left(\frac{1}{x^4}\right)$$

$$= -\frac{5}{8} \ln\left(\frac{2}{x^4}+1\right) + \ln\left(\frac{1}{x^4}+1\right) + C$$

$$= \frac{5}{8} \ln \frac{x^4}{x^4+2} - \ln \frac{x^4}{x^4+1} + C.$$

1908. $\int \frac{x^4 dx}{(x^{10}-10)^2}.$

解 $\int \frac{x^4 dx}{(x^{10}-10)^2} = \frac{1}{5} \int \frac{d(x^5)}{[(x^5-\sqrt{10})(x^5+\sqrt{10})]^2}$

$$\begin{aligned}
&= \frac{1}{200} \int \frac{[(x^5 + \sqrt{10}) - (x^5 - \sqrt{10})]^2}{\{(x^5 - \sqrt{10})(x^5 + \sqrt{10})\}^2} d(x^5) \\
&= \frac{1}{200} \int \left(\frac{1}{x^5 - \sqrt{10}} - \frac{1}{x^5 + \sqrt{10}} \right)^2 d(x^5) \\
&= \frac{1}{200} \int \frac{d(x^5 - \sqrt{10})}{(x^5 - \sqrt{10})^2} - \frac{1}{100} \int \frac{d(x^5)}{(x^5)^2 - 10} \\
&\quad + \frac{1}{200} \int \frac{d(x^5 + \sqrt{10})}{(x^5 + \sqrt{10})^2} \\
&= -\frac{1}{200(x^5 - \sqrt{10})} - \frac{1}{200\sqrt{10}} \ln \left| \frac{x^5 - \sqrt{10}}{x^5 + \sqrt{10}} \right| \\
&\quad - \frac{1}{200(x^5 + \sqrt{10})} + C \\
&= -\frac{1}{100} \left(\frac{x^5}{x^{10} - 10} + \frac{1}{2\sqrt{10}} \ln \left| \frac{x^5 - \sqrt{10}}{x^5 + \sqrt{10}} \right| \right) + C.
\end{aligned}$$

1909. $\int \frac{x^{11} dx}{x^8 + 3x^4 + 2}.$

$$\begin{aligned}
\text{解} \quad \int \frac{x^{11}}{x^8 + 3x^4 + 2} dx &= \frac{1}{4} \int \frac{x^8 d(x^4)}{(x^4 + 1)(x^4 + 2)} \\
&= \frac{1}{4} \int \left[1 - \frac{3x^4 + 2}{(x^4 + 1)(x^4 + 2)} \right] d(x^4) \\
&= \frac{1}{4} \int \left[1 + \frac{1}{x^4 + 1} - \frac{4}{x^4 + 2} \right] d(x^4) \\
&= \frac{x^4}{4} + \frac{1}{4} \ln \frac{x^4 + 1}{(x^4 + 2)^4} + C.
\end{aligned}$$

$$1910. \int \frac{x^9 dx}{(x^{10} + 2x^5 + 2)^2}.$$

$$\begin{aligned} \text{解} \quad & \int \frac{x^9 dx}{(x^{10} + 2x^5 + 2)^2} = \frac{1}{5} \int \frac{x^5 d(x^5)}{[(x^5 + 1)^2 + 1]^2} \\ &= \frac{1}{5} \int \frac{(x^5 + 1) d(x^5 + 1)}{[(x^5 + 1)^2 + 1]^2} - \frac{1}{5} \int \frac{d(x^5 + 1)}{[(x^5 + 1)^2 + 1]^2} \\ &= \frac{1}{10} \int \frac{d[(x^5 + 1)^2 + 1]}{[(x^5 + 1)^2 + 1]^2} - \frac{1}{5} \int \frac{d(x^5 + 1)}{[(x^5 + 1)^2 + 1]^2} \\ &= -\frac{1}{10[(x^5 + 1)^2 + 1]} \\ &\quad - \frac{1}{5} \left\{ \frac{x^5 + 1}{2[(x^5 + 1)^2 + 1]} + \frac{1}{2} \operatorname{arc} \operatorname{tg}(x^5 + 1) \right\}^* + C \\ &= -\frac{x^5 + 2}{10(x^{10} + 2x^5 + 2)} - \frac{1}{10} \operatorname{arc} \operatorname{tg}(x^5 + 1) + C. \end{aligned}$$

*) 利用1817题的结果。

$$1911. \int \frac{x^{2n-1}}{x^n + 1} dx.$$

解 当 $n \neq 0$ 时,

$$\begin{aligned} \int \frac{x^{2n-1}}{x^n + 1} dx &= \int \frac{x^n \cdot x^{n-1} dx}{x^n + 1} \\ &= \frac{1}{n} \int \frac{x^n d(x^n)}{x^n + 1} \\ &= \frac{1}{n} \int \left(1 - \frac{1}{x^n + 1} \right) d(x^n) \end{aligned}$$

$$= \frac{1}{n}(x^n - \ln|x^n + 1|) + C;$$

当 $n=0$ 时,

$$\int \frac{x^{2^n-1}}{x^n+1} dx = \int \frac{dx}{2x} = \frac{1}{2} \ln|x| + C.$$

1912. $\int \frac{x^{3n-1}}{(x^{2n}+1)^2} dx.$

解 当 $n \neq 0$ 时,

$$\begin{aligned} \int \frac{x^{3n-1}}{(x^{2n}+1)^2} dx &= \int \frac{x^{2n} \cdot x^{n-1} dx}{(x^{2n}+1)^2} \\ &= \frac{1}{n} \int \frac{x^{2n} d(x^n)}{(x^{2n}+1)^2} \\ &= \frac{1}{n} \int \frac{(x^{2n}+1) - 1}{(x^{2n}+1)^2} d(x^n) \\ &= \frac{1}{n} \int \frac{d(x^n)}{x^{2n}+1} - \frac{1}{n} \int \frac{d(x^n)}{(x^{2n}+1)^2} \\ &= \frac{1}{n} \operatorname{arc} \operatorname{tg}(x^n) - \frac{1}{n} \left[\frac{x^n}{2(x^{2n}+1)} + \frac{1}{2} \operatorname{arctg}(x^n) \right]^* + C \\ &= \frac{1}{2n} \left[\operatorname{arc} \operatorname{tg}(x^n) - \frac{x^n}{x^{2n}+1} \right] + C. \end{aligned}$$

当 $n=0$ 时,

$$\int \frac{x^{3n-1}}{(x^{2n}+1)^2} dx = \frac{1}{4} \int \frac{dx}{x} = \frac{1}{4} \ln|x| + C.$$

*) 利用 1817 题的结果.

$$1913. \int \frac{dx}{x(x^{10}+2)}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x(x^{10}+2)} &= \frac{1}{2} \int \left(\frac{1}{x} - \frac{x^9}{x^{10}+2} \right) dx \\ &= \frac{1}{2} \ln|x| - \frac{1}{20} \int \frac{d(x^{10}+2)}{x^{10}+2} \\ &= \frac{1}{2} \ln|x| - \frac{1}{20} \ln(x^{10}+2) + C \\ &= \frac{1}{20} \ln \frac{x^{10}}{x^{10}+2} + C. \end{aligned}$$

$$1914. \int \frac{dx}{x(x^{10}+1)^2}.$$

解 由于

$$\begin{aligned} \frac{1}{x(x^{10}+1)^2} &= \frac{x^{10}+1-x^{10}}{x(x^{10}+1)^2} \\ &= \frac{1}{x(x^{10}+1)} - \frac{x^9}{(x^{10}+1)^2} \\ &= \frac{1}{x} - \frac{x^9}{x^{10}+1} - \frac{x^9}{(x^{10}+1)^2}, \end{aligned}$$

所以

$$\begin{aligned} \int \frac{dx}{x(x^{10}+1)^2} &= \int \left[\frac{1}{x} - \frac{x^9}{x^{10}+1} - \frac{x^9}{(x^{10}+1)^2} \right] dx \\ &= \ln|x| - \frac{1}{10} \int \frac{d(x^{10}+1)}{x^{10}+1} - \frac{1}{10} \int \frac{d(x^{10}+1)}{(x^{10}+1)^2} \end{aligned}$$

$$\begin{aligned}
&= \ln|x| - \frac{1}{10} \ln(x^{10} + 1) + \frac{1}{10(x^{10} + 1)} + C \\
&= \frac{1}{10} \ln \frac{x^{10}}{x^{10} + 1} + \frac{1}{10(x^{10} + 1)} + C.
\end{aligned}$$

1915. $\int \frac{1-x^7}{x(1+x^7)} dx.$

$$\begin{aligned}
\text{解} \quad &\int \frac{1-x^7}{x(1+x^7)} dx = \int \left(\frac{1}{x} - \frac{2x^6}{1+x^7} \right) dx \\
&= \ln|x| - \frac{2}{7} \int \frac{d(1+x^7)}{1+x^7} \\
&= \ln|x| - \frac{2}{7} \ln|1+x^7| + C \\
&= \frac{1}{7} \ln \frac{|x|^7}{(1+x^7)^2} + C.
\end{aligned}$$

1916. $\int \frac{x^4-1}{x(x^4-5)(x^5-5x+1)} dx.$

$$\begin{aligned}
\text{解} \quad &\int \frac{x^4-1}{x(x^4-5)(x^5-5x+1)} dx \\
&= \frac{1}{5} \int \frac{d(x^5-5x)}{(x^5-5x)(x^5-5x+1)} \\
&= \frac{1}{5} \int \left(\frac{1}{x^5-5x} - \frac{1}{x^5-5x+1} \right) d(x^5-5x) \\
&= \frac{1}{5} \int \frac{d(x^5-5x)}{x^5-5x} - \frac{1}{5} \int \frac{d(x^5-5x+1)}{x^5-5x+1}
\end{aligned}$$

$$= \frac{1}{5} \ln \left| \frac{x(x^4 - 5)}{x^5 - 5x + 1} \right| + C.$$

1917. $\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx.$

解 由于

$$\begin{aligned} \frac{x^2 + 1}{x^4 + x^2 + 1} &= \frac{x^2 + 1}{(x^2 + 1)^2 - x^2} \\ &= \frac{x^2 + 1}{(x^2 - x + 1)(x^2 + x + 1)} \\ &= \frac{1}{2} \left(\frac{1}{x^2 - x + 1} + \frac{1}{x^2 + x + 1} \right), \end{aligned}$$

所以

$$\begin{aligned} \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx &= \frac{1}{2} \int \frac{dx}{x^2 - x + 1} + \frac{1}{2} \int \frac{dx}{x^2 + x + 1} \\ &= \frac{1}{2} \int \frac{d\left(x - \frac{1}{2}\right)}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x - 1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x + 1}{\sqrt{3}} + C_1 \\ &= \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x^2 - 1}{x\sqrt{3}} + C. \end{aligned}$$

1918. $\int \frac{x^2 - 1}{x^4 + x + x^2 + x + 1} dx.$

$$\begin{aligned}
& \text{解} \quad \int \frac{x^2 - 1}{x^4 + x^3 + x^2 + x + 1} dx \\
&= \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1} \\
&= \int \frac{d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1} = \int \frac{d\left(x + \frac{1}{x} + \frac{1}{2}\right)}{\left[\left(x + \frac{1}{x}\right) + \frac{1}{2}\right]^2 - \frac{5}{4}} \\
&= \frac{1}{\sqrt{5}} \ln \frac{x + \frac{1}{x} + \frac{1}{2} - \frac{\sqrt{5}}{2}}{x + \frac{1}{x} + \frac{1}{2} + \frac{\sqrt{5}}{2}} + C \\
&= \frac{1}{\sqrt{5}} \ln \frac{2x^2 + (1 - \sqrt{5})x + 2}{2x^2 + (1 + \sqrt{5})x + 2} + C.
\end{aligned}$$

1919. $\int \frac{x^5 - x}{x^6 + 1} dx.$

$$\begin{aligned}
& \text{解} \quad \int \frac{x^5 - x}{x^6 + 1} dx = \frac{1}{2} \int \frac{(x^2)^2 - 1}{(x^2)^4 + 1} d(x^2) \\
&= \frac{1}{4\sqrt{2}} \ln \frac{x^4 - x^2\sqrt{2} + 1^{*})}{x^4 + x^2\sqrt{2} + 1} + C.
\end{aligned}$$

*) 利用1713题的结果。

1920. $\int \frac{x^4 + 1}{x^6 + 1} dx.$

$$\text{解} \quad \int \frac{x^4 + 1}{x^6 + 1} dx = \int \frac{(x^4 - x^2 + 1) + x^2}{x^6 + 1} dx$$

$$\begin{aligned}
&= \int \frac{x^4 - x^2 + 1}{(x^2 + 1)(x^4 - x^2 + 1)^2} dx + \int \frac{x^2 dx}{x^6 + 1} \\
&= \int \frac{1}{x^2 + 1} dx + \frac{1}{3} \int \frac{d(x^3)}{(x^3)^2 + 1} \\
&= \operatorname{arctg} x + \frac{1}{3} \operatorname{arctg}(x^3) + C.
\end{aligned}$$

1921. 试导出计算积分

$$I_n = \int \frac{dx}{(ax^2 + bx + c)^n} \quad (a \neq 0)$$

的递推公式.

利用这个公式计算

$$I_3 = \int \frac{dx}{(x^2 + x + 1)^3}.$$

解 由于

$$4a(ax^2 + bx + c) = (2ax + b)^2 + (4ac - b^2) = t^2 + \Delta,$$

其中 $t = 2ax + b$, $\Delta = 4ac - b^2$. 于是

$$\begin{aligned}
I_n &= \int \frac{dx}{(ax^2 + bx + c)^n} = \int \frac{(4a)^n dx}{[(2ax + b)^2 + \Delta]^n} \\
&= 2^{2n-1} a^{n-1} \int \frac{dt}{(t^2 + \Delta)^n}.
\end{aligned}$$

当 $\Delta \neq 0$ 时, 对于积分 $\int \frac{dt}{(t^2 + \Delta)^n}$ 施用分部积分法,

即有

$$\int \frac{dt}{(t^2 + \Delta)^n} = \frac{t}{(t^2 + \Delta)^n} + 2n \int \frac{t^2 dt}{(t^2 + \Delta)^{n+1}}$$

$$\begin{aligned}
&= \frac{t}{(t^2 + \Delta)^n} + 2n \int \frac{(t^2 + \Delta) - \Delta}{(t^2 + \Delta)^{n+1}} dt \\
&= \frac{t}{(t^2 + \Delta)^n} + 2n \int \frac{dt}{(t^2 + \Delta)^n} - 2n \Delta \int \frac{dt}{(t^2 + \Delta)^{n+1}}.
\end{aligned}$$

若令 $\bar{I}_n = \int \frac{dt}{(t^2 + \Delta)^n}$, 则得

$$\bar{I}_n = \frac{t}{(t^2 + \Delta)^n} + 2n \bar{I}_n - 2n \Delta \bar{I}_{n+1},$$

$$\text{或 } \bar{I}_{n+1} = \frac{1}{2n \Delta} \cdot \frac{t}{(t^2 + \Delta)^n} + \frac{2n-1}{2n} \cdot \frac{1}{\Delta} \cdot \bar{I}_n,$$

$$\text{从而 } \bar{I}_n = \frac{1}{2(n-1)\Delta} \cdot \frac{t}{(t^2 + \Delta)^{n-1}} + \frac{2n-3}{2n-2} \cdot \frac{1}{\Delta} \bar{I}_{n-1}.$$

代入 I_n , 即得

$$\begin{aligned}
I_n &= 2^{2n-1} \cdot a^{n-1} \cdot \left\{ \frac{1}{2(n-1)\Delta} \cdot \frac{t}{(t^2 + \Delta)^{n-1}} \right. \\
&\quad \left. + \frac{2n-3}{2n-2} \cdot \frac{1}{\Delta} \cdot \bar{I}_{n-1} \right\} \\
&= 2^{2n-1} \cdot a^{n-1} \cdot \left\{ \frac{1}{2(n-1)\Delta} \cdot \frac{2ax+b}{(4a)^{n-1}(ax^2+bx+c)^{n-1}} \right. \\
&\quad \left. + \frac{2n-3}{2n-2} \cdot \frac{1}{\Delta} \cdot \frac{2a}{(4a)^{n-1}} \int \frac{dx}{(ax^2+bx+c)^{n-1}} \right\} \\
&= \frac{1}{(n-1)\Delta} \cdot \frac{2ax+b}{(ax^2+bx+c)^{n-1}} + \frac{2n-3}{n-1} \cdot \frac{2a}{\Delta} I_{n-1},
\end{aligned}$$

最后得递推公式

$$I_n = \frac{1}{(n-1)\Delta} \cdot \frac{2ax+b}{(ax^2+bx+c)^{n-1}} + \frac{2n-3}{n-1} \cdot \frac{2a}{\Delta} I_{n-1}.$$

当 $\Delta = 0$ 时, 则有

$$\begin{aligned} I_n &= \int \frac{(4a)^n dx}{(2ax+b)^{2n}} = 2^{2n-1} \cdot a^{n-1} \int \frac{d(2ax+b)}{(2ax+b)^{2n}} \\ &= \frac{1}{a^n(1-2n)} \left(x + \frac{b}{2a}\right)^{1-2n} + C. \end{aligned}$$

对于 I_3 , $\Delta \neq 0$, 两次运用上述递推公式, 即得

$$\begin{aligned} I_3 &= \int \frac{dx}{(x^2+x+1)^3} = \frac{2x+1}{6(x^2+x+1)^2} + \int \frac{dx}{(x^2+x+1)^2} \\ &= \frac{2x+1}{6(x^2+x+1)^2} + \frac{2x+1}{3(x^2+x+1)} + \frac{2}{3} \int \frac{dx}{x^2+x+1} \\ &= \frac{2x+1}{6(x^2+x+1)^2} + \frac{2x+1}{3(x^2+x+1)} \\ &\quad + \frac{4}{3\sqrt{3}} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}}\right) + C. \end{aligned}$$

1922. 利用代换 $t = \frac{x+a}{x+b}$ 来计算积分:

$$I = \int \frac{dx}{(x+a)^m(x+b)^n}.$$

(m 及 n 为自然数).

利用这个代换, 求

$$\int \frac{dx}{(x-2)^2(x+3)^3}.$$

解 设 $t = \frac{x+a}{x+b}$, 则 $1-t = \frac{b-a}{x+b}$ 或 $x+b = \frac{b-a}{1-t}$,

$$dt = \frac{b-a}{(x+b)^2} dx = \frac{(1-t)^2}{b-a} dx \text{ 或 } dx = \frac{b-a}{(1-t)^2} dt,$$

$$\text{及 } x+a = t(x+b) = \frac{t(b-a)}{1-t}.$$

代入 I , 即得

$$I = \frac{1}{(b-a)^{m+n-1}} \int \frac{(1-t)^{m+n-2}}{t^n} dt \quad (a \neq b).$$

将 $(1-t)^{m+n-2}$ 展开, 即可分项积分求得 I .

如果 $b=a$, 则

$$I = \int \frac{dx}{(x+a)^{m+n}} = \frac{1}{1-m-n} (x+a)^{1-m-n} + C.$$

令 $a=-2, b=3, m=2$ 及 $n=3$, 并设 $t = \frac{x-2}{x+3}$,

即得

$$\begin{aligned} \int \frac{dx}{(x-2)^2(x+3)^3} &= \frac{1}{5^4} \int \frac{(1-t)^3}{t^2} dt \\ &= \frac{1}{5^4} \int \left(\frac{1}{t^2} - \frac{3}{t} + 3 - t \right) dt \\ &= \frac{1}{625} \left(-\frac{1}{t} - 3 \ln |t| + 3t - \frac{t^2}{2} \right) + C \\ &= \frac{1}{625} \left[-\frac{x+3}{x-2} - 3 \ln \left| \frac{x-2}{x+3} \right| + \frac{3(x-2)}{x+3} \right] \end{aligned}$$

$$\left. -\frac{(x-2)^2}{2(x+3)^2} \right\} + C.$$

1923. 若 $P_n(x)$ 为 x 的 n 次多项式, 计算

$$\int \frac{P_n(x)}{(x-a)^{n+1}} dx.$$

解 由于 $P_n(x)$ 为 x 的 n 次多项式, 故得

$$P_n(x) = \sum_{k=0}^n \frac{P_n^{(k)}(a)}{k!} (x-a)^k,$$

其中 $P_n^{(0)}(a) = P_n(a)$, $0! = 1$.

于是,

$$\begin{aligned} \int \frac{P_n(x)}{(x-a)^{n+1}} dx &= \sum_{k=0}^{n-1} \frac{1}{k!} P_n^{(k)}(a) \int \frac{dx}{(x-a)^{n-k+1}} \\ &+ \frac{1}{n!} P_n^{(n)}(a) \int \frac{dx}{x-a} \\ &= - \sum_{k=0}^{n-1} \frac{P_n^{(k)}(a)}{k!(n-k)(x-a)^{n-k}} \\ &+ \frac{1}{n!} P_n^{(n)}(a) \ln|x-a| + C, \end{aligned}$$

其中 $\frac{P_n^{(n)}(a)}{n!} = a_0$ 为 $P_n(x)$ 的首项系数, 即

$$\begin{aligned} P_n(x) &= a_0(x-a)^n + a_1(x-a)^{n-1} + \dots \\ &+ a_{n-1}(x-a) + a_n. \end{aligned}$$

1924⁺. 设 $R(x) = R^*(x^2)$, 其中 R^* 为有理函数, 则函数 $R(x)$ 分解为有理分式时有甚么特性?

解 设 $R^*(x) = P(x) + H(x)$,

其中 $P(x)$ 是多项式; 若 $R^*(x)$ 本身也为多项式时, 则 $H(x) \equiv 0$; 否则 $H(x) = \frac{P_1(x)}{Q_1(x)}$ 是真分式, 而 $P_1(x)$, $Q_1(x)$ 也均为多项式.

设 $Q_1(x)$ 有非负实根为 a_i^2 , 其重数为 α_i ($i=1, 2, \dots, m$); 负根为 $-b_k^2$, 其重数为 β_k ($k=1, 2, \dots, t$); 二次因式为 $x^2 + C_p x + D_p$, 其重数为 ν_p ($p=1, 2, \dots, s$). 其中 $C_p^2 - 4D_p < 0$, 于是,

$$Q_1(x) = \begin{cases} a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i} \cdot \prod_{k=1}^t (x + b_k^2)^{\beta_k} \cdot \prod_{p=1}^s (x^2 + C_p x + D_p)^{\nu_p}, & \text{当 } m \neq 0, t \neq 0, s \neq 0 \text{ 时;} \\ a_0 \prod_{k=1}^t (x + b_k^2)^{\beta_k} \cdot \prod_{p=1}^s (x^2 + C_p x + D_p)^{\nu_p}, & \text{当 } m = 0, \\ & t \neq 0, s \neq 0 \text{ 时;} \\ a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i} \cdot \prod_{p=1}^s (x^2 + C_p x + D_p)^{\nu_p}, & \text{当 } m \neq 0, \\ & t = 0, s \neq 0 \text{ 时;} \\ a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i} \cdot \prod_{k=1}^t (x + b_k^2)^{\beta_k}, & \text{当 } m \neq 0, t \neq 0, \\ & s = 0 \text{ 时;} \\ a_0 \prod_{i=1}^m (x - a_i^2)^{\alpha_i}, & \text{当 } m \neq 0, t = 0, s = 0 \text{ 时;} \\ a_0 \prod_{k=1}^t (x + b_k^2)^{\beta_k}, & \text{当 } m = 0, t \neq 0, s = 0 \text{ 时;} \\ a_0 \prod_{p=1}^s (x^2 + C_p x + D_p)^{\nu_p}, & \text{当 } m = 0, t = 0, s \neq 0 \text{ 时;} \end{cases}$$

以下就 $Q_1(x)$ 表达式中的第一种情形予以论证。

由 $C_p^2 - 4D_p < 0$, 必有

$$x^4 + C_p x^2 + D_p = (x^2 + E_p x + F_p) \cdot (x^2 - E_p x + F_p)$$

($p = 1, 2, \dots, s$), 则此时有

$$Q_1(x^2) = a_0 \prod_{i=1}^m (x - a_i)^{\alpha_i} (x + a_i)^{\alpha_i} \cdot \prod_{k=1}^t (x^2 + b_k^2)^{\beta_k}$$

$$\cdot \prod_{p=1}^s (x^2 + E_p x + F_p)^{\gamma_p} (x^2 - E_p x + F_p)^{\gamma_p}, \text{ 以及}$$

$$H(x^2) = \frac{P_1(x^2)}{Q_1(x^2)} = \sum_{i=1}^m \sum_{l=1}^{\alpha_i} \left[\frac{A_{il}}{(a_i - x)^l} + \frac{A'_{il}}{(a_i + x)^l} \right] \\ + \sum_{k=1}^t \sum_{l=1}^{\beta_k} \frac{B_{kl}x + C_{kl}}{(x^2 + b_k^2)^l} + \sum_{p=1}^s \sum_{l=1}^{\gamma_p} \left[\frac{M_{pl}x + N_{pl}}{(x^2 + E_p x + F_p)^l} \right. \\ \left. + \frac{M'_{pl}x + N'_{pl}}{(x^2 - E_p x + F_p)^l} \right].$$

显然有 $H(x^2) = H((-x)^2)$, 由 $H(x^2)$ 的分解式的唯一性, 比较系数, 即得常数关系为:

$$A'_{i1l} = A_{i1l}, M'_{p12} = -M_{p12}, N'_{p12} = N_{p12}, B_{kl3} = 0, \\ (l_1 = 1, 2, \dots, \alpha_i, i = 1, 2, \dots, m; l_2 = 1, 2, \dots, \beta_k, \\ p = 1, 2, \dots, s; l_3 = 1, 2, \dots, \beta_k, k = 1, 2, \dots, t). \text{ 最后得}$$

$$R(x) = P(x^2) + H(x^2) = \\ = P(x^2) + \sum_{i=1}^m \sum_{l=1}^{\alpha_i} A_{il} \left[\frac{1}{(a_i - x)^l} + \frac{1}{(a_i + x)^l} \right] \\ + \sum_{k=1}^t \sum_{l=1}^{\beta_k} \frac{C_{kl}}{(x^2 + b_k^2)^l} + \sum_{p=1}^s \sum_{l=1}^{\gamma_p} \left[\frac{M_{pl}x + N_{pl}}{(x^2 + E_p x + F_p)^l} \right. \\ \left. - \frac{M_{pl}x - N_{pl}}{(x^2 - E_p x + F_p)^l} \right].$$

如若 $H(x) \neq 0$, 而 $m=0$, 但 $t \neq 0$, $s \neq 0$ 时, 则在上述表达式中就应该缺乏第二项的和式, 形如

$$R(x) = P(x^2) + \sum_{k=1}^t \sum_{l=1}^{t_k} + \sum_{j=1}^s \sum_{l=1}^{s_j},$$

其它情形可以类似推演, 此处不再一一细叙. 至于当 $H(x) \equiv 0$ 时, 当然有 $R(x) = P(x^2)$.

另外, 本题也可在复数域上作分解考虑.

仍记 $R^*(x) = P(x) + H(x)$, 其中 $P(x)$ 为多项式, 而 $H(x)$ 要么是零 (当 $R^*(x)$ 为多项式时), 要么是一个真分式. 例如 $H(x) \neq 0$ 时, 记 $H(x) = \frac{P_1(x)}{Q_1(x)}$ 是其真分式. $P_1(x), Q_1(x)$ 为多项式. 若记 $Q_1(x)$ 在复数域中的根为 α_i , 其相应重数记为 n_i ($i = 1, 2, \dots, m$; 显然 $m \geq 1$). 即

$$Q_1(x) = a_0 \prod_{i=1}^m (x - \alpha_i)^{n_i},$$

那么 $Q_1(x^2)$ 中的每一项 $x^2 - \alpha_i$ 可分解为一次式乘积

$$x^2 - \alpha_i = (x - b_i)(x + b_i),$$

于是

$$Q_1(x^2) = a_0 \prod_{i=1}^m (x - b_i)^{n_i} (x + b_i)^{n_i}.$$

相应地有

$$\begin{aligned} H(x^2) &= \frac{P_1(x^2)}{Q_1(x^2)} = \sum_{i=1}^m \sum_{k=1}^{n_i} \left[\frac{B_{ik}}{(x - b_i)^k} + \frac{B'_{ik}}{(x + b_i)^k} \right], \\ &= \sum_{i=1}^m \sum_{k=1}^{n_i} \left[\frac{A_{ik}}{(x - b_i)^k} + \frac{A'_{ik}}{(x + b_i)^k} \right]. \end{aligned}$$

由 $H(x^2) = H((-x)^2)$, 从 $H(x^2)$ 的分解式的唯一

性, 比较系数, 即得 $A'_{ik} = A_{ik} (k=1, 2, \dots, n_i; i=1, 2, \dots, m)$. 最后得到

$$R(x) = P(x^2) + H(x^2) = P(x^2) + \sum_{i=1}^m \sum_{k=1}^{n_i} \left[\frac{A_{ik}}{(b_i - x)^k} + \frac{A_{ik}}{(b_i + x)^k} \right],$$

其中 b_i 为分母 $Q_1(x^2)$ 的根, A_{ik} 为常数.

1925. 计算

$$\int \frac{dx}{1+x^{2n}}$$

式中 n 为正整数.

解 先将被积函数分解成部分分式之和, 我们可以证明

$$\frac{1}{1+x^{2n}} = \frac{1}{n} \sum_{k=1}^n \frac{1 - x \cos \frac{2k-1}{2n} \pi}{x^2 - 2x \cos \frac{2k-1}{2n} \pi + 1}.$$

事实上, 记多项式 $x^{2n} + 1$ 的 $2n$ 个根为 $a_k (k=1, 2, \dots, 2n)$, 显然 $a_k = \cos \frac{2k-1}{2n} \pi + j \sin \frac{2k-1}{2n} \pi$, 其中 $j = \sqrt{-1}$ 为虚数单位.

于是, $|a_k| = 1, a_k^{2n} = -1, \bar{a}_k = a_{2n-k+1}$,

$$a_k \bar{a}_k = 1, a_k + \bar{a}_k = 2 \cos \frac{2k-1}{2n} \pi.$$

设
$$\frac{1}{1+x^{2n}} = \sum_{k=1}^{2n} \frac{A_k}{x-a_k},$$

即
$$1 = \sum_{k=1}^{2n} \frac{A_k(1+x^{2n})}{x-a_k}$$

令 $x \rightarrow \alpha_i$, 并应用洛比塔法则, 即得

$$\begin{aligned} 1 &= \lim_{x \rightarrow \alpha_i} \sum_{k=1}^{2n} \frac{A_k(1+x^{2n})}{x-\alpha_k} = \lim_{x \rightarrow \alpha_i} \frac{A_i(1+x^{2n})}{x-\alpha_i} \\ &= \lim_{x \rightarrow \alpha_i} (2n A_i x^{2n-1}) \\ &= 2n A_i \cdot \frac{\alpha_i^{2n}}{\alpha_i} = -\frac{2n A_i}{\alpha_i} \quad (i=1, 2, \dots, 2n), \end{aligned}$$

即 $A_k = -\frac{\alpha_k}{2n} \quad (k=1, 2, \dots, 2n)$.

于是,

$$\begin{aligned} \frac{1}{1+x^{2n}} &= -\frac{1}{2n} \sum_{k=1}^{2n} \frac{\alpha_k}{x-\alpha_k} \\ &= -\frac{1}{2n} \sum_{k=1}^n \left(\frac{\alpha_k}{x-\alpha_k} + \frac{\bar{\alpha}_k}{x-\bar{\alpha}_k} \right) \\ &= -\frac{1}{2n} \sum_{k=1}^n \frac{(\alpha_k + \bar{\alpha}_k)x - 2\alpha_k \bar{\alpha}_k}{x^2 - (\alpha_k + \bar{\alpha}_k)x + \alpha_k \bar{\alpha}_k} \\ &= \frac{1}{n} \sum_{k=1}^n \frac{1 - x \cos \frac{2k-1}{2n} \pi}{x^2 - 2x \cos \frac{2k-1}{2n} \pi + 1} \end{aligned}$$

最后得到

$$\int \frac{dx}{1+x^{2n}} = \frac{1}{n} \sum_{k=1}^n \int \frac{1 - x \cos \frac{2k-1}{2n} \pi}{x^2 - 2x \cos \frac{2k-1}{2n} \pi + 1} dx$$

$$\begin{aligned}
&= -\frac{1}{2n} \sum_{k=1}^n \left[\cos \frac{2k-1}{2n} \pi \cdot \int \frac{2x - 2\cos \frac{2k-1}{2n} \pi}{x^2 - 2x\cos \frac{2k-1}{2n} \pi + 1} dx \right] \\
&+ \frac{1}{n} \sum_{k=1}^n \left[\sin^2 \frac{2k-1}{2n} \pi \cdot \int \frac{dx}{\left(x - \cos \frac{2k-1}{2n} \pi\right)^2 + \sin^2 \frac{2k-1}{2n} \pi} \right] \\
&= -\frac{1}{2n} \sum_{k=1}^n \left[\cos \frac{2k-1}{2n} \pi \cdot \ln \left(x^2 - 2x\cos \frac{2k-1}{2n} \pi + 1 \right) \right] \\
&+ \frac{1}{n} \sum_{k=1}^n \left[\sin \frac{2k-1}{2n} \pi \cdot \operatorname{arctg} \frac{x - \cos \frac{2k-1}{2n} \pi}{\sin \frac{2k-1}{2n} \pi} \right] + C.
\end{aligned}$$

§3. 无理函数的积分法

化被积函数为有理函数，以求下列积分：

1926. $\int \frac{dx}{1 + \sqrt{x}}.$

解 设 $\sqrt{x} = t$ ，则 $x = t^2$ ， $dx = 2t dt$ 。

代入得

$$\begin{aligned}
\int \frac{dx}{1 + \sqrt{x}} &= 2 \int \frac{t dt}{1+t} = 2 \int \left(1 - \frac{1}{1+t} \right) dt \\
&= 2[t - \ln(1+t)] + C = 2\sqrt{x} - 2\ln(1 + \sqrt{x}) + C.
\end{aligned}$$

1927. $\int \frac{dx}{x(1 + 2\sqrt{x} + \sqrt[3]{x})}.$

解 设 $\sqrt[6]{x} = t$, 则 $x = t^6$, $dx = 6t^5 dt$.

代入得

$$\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})} = 6 \int \frac{dt}{t(1+2t^3+t^2)}$$

$$= 6 \int \frac{dt}{t(1+t)(2t^2-t+1)}$$

$$= 6 \int \left[\frac{1}{t} - \frac{1}{4(1+t)} - \frac{6t-1}{4(2t^2-t+1)} \right] dt$$

$$= 6 \left\{ \ln t - \frac{1}{4} \ln(1+t) - \frac{3}{8} \int \frac{4t-1}{2t^2-t+1} dt \right.$$

$$\left. - \frac{1}{16} \int \frac{d\left(t - \frac{1}{4}\right)}{\left(t - \frac{1}{4}\right)^2 + \frac{7}{16}} \right\}$$

$$= 6 \left\{ \ln|t| - \frac{1}{4} \ln|1+t| - \frac{3}{8} \ln(2t^2-t+1) \right.$$

$$\left. - \frac{1}{4\sqrt{7}} \operatorname{arctg} \frac{4t-1}{\sqrt{7}} \right\} + C$$

$$= \frac{3}{4} \ln \frac{t^6}{(1+t)^2(2t^2-t+1)^3}$$

$$- \frac{3}{2\sqrt{7}} \operatorname{arctg} \frac{4t-1}{\sqrt{7}} + C$$

$$= \frac{3}{4} \ln \frac{x \cdot \sqrt[3]{x}}{(1+\sqrt[6]{x})^2(2\sqrt[3]{x}-\sqrt[6]{x}+1)^3}$$

$$- \frac{3}{2\sqrt{7}} \operatorname{arctg} \frac{4\sqrt[6]{x}-1}{\sqrt{7}} + C.$$

$$1928^+ \int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}} dx.$$

解 设 $\sqrt[3]{2+x}=t$, 则 $x=t^3-2$, $dx=3t^2 dt$.
代入得

$$\begin{aligned} \int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}} dx &= 3 \int \frac{t^6-2t^3}{t^3+t-2} dt \\ &= 3 \int \left(t^3-t + \frac{t^2-2t}{t^3+t-2} \right) dt \\ &= \frac{3}{4}t^4 - \frac{3}{2}t^2 + 3 \int \left[-\frac{1}{4(t-1)} + \frac{\frac{5}{4}t - \frac{1}{2}}{t^2+t+2} \right] dt \\ &= \frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4} \ln|t-1| \\ &\quad + \frac{15}{8} \int \frac{2t+1}{t^2+t+2} dt - \frac{27}{8} \int \frac{d\left(t + \frac{1}{2}\right)}{\left(t + \frac{1}{2}\right)^2 + \frac{7}{4}} \\ &= \frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4} \ln|t-1| + \frac{15}{8} \ln(t^2+t+2) \\ &\quad - \frac{27}{4\sqrt{7}} \operatorname{arc} \operatorname{tg} \left(\frac{2t+1}{\sqrt{7}} \right) + C, \end{aligned}$$

其中 $t = \sqrt[3]{2+x}$.

$$1929. \int \frac{1-\sqrt{x-1}}{1+\sqrt[3]{x+1}} dx.$$

解 设 $\sqrt[6]{x+1}=t$, 则 $x=t^6-1$, $dx=6t^5 dt$.

代入得

$$\begin{aligned}\int \frac{1 - \sqrt{x+1}}{1 + \sqrt[3]{x+1}} dx &= 6 \int \frac{t^5(1-t^3)}{1+t^2} dt \\ &= 6 \int \left(-t^6 + t^4 + t^3 - t^2 - t + 1 + \frac{t-1}{1+t^2} \right) dt \\ &= -\frac{6}{7}t^7 + \frac{6}{5}t^5 + \frac{3}{2}t^4 - 2t^3 - 3t^2 + 6t \\ &\quad + 3 \ln(1+t^2) - 6 \operatorname{arc} \operatorname{tg} t + C,\end{aligned}$$

其中 $t = \sqrt[3]{x+1}$.

1930. $\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3}$.

解 设 $\sqrt[4]{x} = t$, 则 $x = t^4$, $dx = 4t^3 dt$.

代入得

$$\begin{aligned}\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3} &= 4 \int \frac{t dt}{(1+t)^3} \\ &= 4 \int \left[\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3} \right] dt \\ &= -\frac{4}{1+t} + \frac{2}{(1+t)^2} + C \\ &= \frac{2}{(1+\sqrt[4]{x})^2} - \frac{4}{1+\sqrt[4]{x}} + C.\end{aligned}$$

1931. $\int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx$.

解 设 $\sqrt{\frac{x+1}{x-1}}=t$, 则

$$x = \frac{t^2 + 1}{t^2 - 1}, \quad dx = -\frac{4t}{(t^2 - 1)^2} dt.$$

代入得

$$\begin{aligned} \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx &= \int \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} dx \\ &= -4 \int \frac{t dt}{(t-1)(t+1)^3} \\ &= \int \left[-\frac{2}{(t+1)^3} + \frac{1}{(t+1)^2} + \frac{1}{2(t+1)} - \frac{1}{2(t-1)} \right] dt \\ &= \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{2} \ln \left| \frac{t+1}{t-1} \right| + C_1 \\ &= \frac{1}{2} x^2 - \frac{1}{2} x \sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C. \end{aligned}$$

如果不限制将被积函数化为有理函数, 本题的解法可简单些. 事实上,

$$\begin{aligned} \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx &= \int \frac{(\sqrt{x+1} - \sqrt{x-1})^2}{(x+1) - (x-1)} dx \\ &= \int (x - \sqrt{x^2 - 1}) dx \\ &= \frac{1}{2} x^2 - \frac{1}{2} x \sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C. \end{aligned}$$

$$1932. \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$$

解 设 $\sqrt[3]{\frac{x+1}{x-1}} = t$, 则

$$x = \frac{t^3 + 1}{t^3 - 1}, \quad dx = -\frac{6t^2}{(t^3 - 1)^2} dt.$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} &= -\frac{3}{2} \int dt = -\frac{3}{2}t + C \\ &= -\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + C. \end{aligned}$$

$$1933. \int \frac{x dx}{\sqrt[4]{x^3(a-x)}} \quad (a > 0).$$

解 设 $\sqrt[4]{\frac{a-x}{x}} = t$, 则 $x = \frac{a}{1+t^4}$,

$$dx = -\frac{4at^3}{(1+t^4)^2} dt.$$

代入得

$$\begin{aligned} \int \frac{x dx}{\sqrt[4]{x^3(a-x)}} &= \int \frac{dx}{\sqrt[4]{\frac{a-x}{x}}} = -4a \int \frac{t^2}{(1+t^4)^2} dt \\ &= -4a \int \left[\frac{t}{(t^2 - t\sqrt{2} + 1)(t^2 + t\sqrt{2} + 1)} \right]^2 dt \\ &= -\frac{a}{2} \int \left(\frac{1}{t^2 - t\sqrt{2} + 1} - \frac{1}{t^2 + t\sqrt{2} + 1} \right)^2 dt \end{aligned}$$

$$= -\frac{a}{2} \int \frac{dt}{(t^2 - t\sqrt{2} + 1)^2} - \frac{a}{2} \int \frac{dt}{(t^2 + t\sqrt{2} + 1)^2} \\ + a \int \frac{dt}{t^2 + 1}.$$

现在分别求上述积分，利用1921题的递推公式，即得

$$\int \frac{dt}{(t^2 - t\sqrt{2} + 1)^2} = \frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} \\ + \int \frac{dt}{t^2 - t\sqrt{2} + 1} \\ = \frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} + \int \frac{d\left(t - \frac{\sqrt{2}}{2}\right)}{\left(t - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \\ = \frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} + \sqrt{2} \operatorname{arc} \operatorname{tg}(\sqrt{2}t - 1) + C_1$$

及

$$\int \frac{dt}{(t^2 + t\sqrt{2} + 1)^2} = \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} \\ + \int \frac{dt}{t^2 + t\sqrt{2} + 1} \\ = \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} + \int \frac{d\left(t + \frac{\sqrt{2}}{2}\right)}{\left(t + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}$$

$$= \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} + \sqrt{2} \operatorname{arctg}(\sqrt{2}t + 1) + C_2.$$

利用1884题的结果, 即得

$$\int \frac{dt}{t^4 + 1} = \frac{1}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\ + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t\sqrt{2}}{1 - t^2} + C_3.$$

最后得到

$$\int \frac{x dx}{\sqrt{x^3(a-x)}} = -\frac{a}{2} \left[\frac{2t - \sqrt{2}}{2(t^2 - t\sqrt{2} + 1)} \right. \\ \left. + \frac{2t + \sqrt{2}}{2(t^2 + t\sqrt{2} + 1)} \right] - \frac{a\sqrt{2}}{2} \left[\operatorname{arctg}(\sqrt{2}t - 1) \right. \\ \left. + \operatorname{arctg}(\sqrt{2}t + 1) \right] + \frac{a}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\ + \frac{a}{2\sqrt{2}} \operatorname{arctg} \left(\frac{t\sqrt{2}}{1 - t^2} \right) + C_4 \\ = -\frac{at^3}{1 + t^4} + \frac{a}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\ - \frac{a}{2\sqrt{2}} \operatorname{arctg} \left(\frac{t\sqrt{2}}{1 - t^2} \right) + C_4 \\ = -\frac{at^3}{1 + t^4} + \frac{a}{4\sqrt{2}} \ln \frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1} \\ + \frac{a}{2\sqrt{2}} \operatorname{arctg} \left(\frac{1 - t^2}{t\sqrt{2}} \right) + C,$$

其中 $t = \sqrt[n]{\frac{a-x}{x}}$ ($0 < x < a$).

1934. $\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}$ (n 为自然数).

解 当 $a=b$ 时, 显然被积函数为 $(x-a)^{-2}$, 因此积分为 $-\frac{1}{x-a} + C$; 当 $a \neq b$ 时, 设 $\sqrt[n]{\frac{x-b}{x-a}} = t$, 则

$$x = a + \frac{a-b}{t^n - 1}, \quad dx = -\frac{n(a-b)t^{n-1}}{(t^n - 1)^2} dt,$$

$$x-a = \frac{a-b}{t^n - 1}, \quad x-b = \frac{(a-b)t^n}{t^n - 1},$$

代入得

$$\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} = -\frac{n}{a-b} \int dt$$

$$= -\frac{n}{a-b} t + C$$

$$= -\frac{n}{a-b} \sqrt[n]{\frac{x-b}{x-a}} + C.$$

1935. $\int \frac{dx}{1 + \sqrt{x} + \sqrt{1+x}}$.

解 设 $\sqrt{x} = \frac{t^2 - 1}{2t}$ 并限制 $t > 1$, 则

$$x = \left(\frac{t^2 - 1}{2t}\right)^2, \quad dx = -\frac{t^4 - 1}{2t^3} dt, \quad \sqrt{x+1} = \frac{t^2 + 1}{2t},$$

$$t = \sqrt{x} + \sqrt{x+1}.$$

代入得

$$\begin{aligned}\int \frac{dx}{1 + \sqrt{x} + \sqrt{x+1}} &= \frac{1}{2} \int \frac{t^4 - 1}{t^3(t+1)} dt \\ &= \frac{1}{2} \int \left(1 - \frac{1}{t} + \frac{1}{t^2} - \frac{1}{t^3}\right) dt \\ &= \frac{1}{2} \left(t - \ln t - \frac{1}{t} + \frac{1}{2t^2}\right) + C_1 \\ &= \sqrt{x} - \frac{1}{2} \ln(\sqrt{x} + \sqrt{x+1}) \\ &\quad + \frac{x}{2} - \frac{1}{2} \sqrt{x(x+1)} + C.\end{aligned}$$

1936. 证明: 若

$$p + q = kn,$$

式中 k 为整数, 则积分

$$\int R\left[x, (x-a)^{\frac{p}{n}}(x-b)^{\frac{q}{n}}\right] dx$$

(式中 R 为有理函数及 p, q, n 为整数) 为初等函数.

证 当 $a=b$ 时, $(x-a)^{\frac{p}{n}}(x-b)^{\frac{q}{n}} = (x-a)^k$, 则积分显然为初等函数.

当 $a \neq b$ 时, 设 $\frac{x-a}{x-b} = y (\neq 1)$, 则

$$x = \frac{a-by}{1-y}, \quad dx = \frac{a-b}{(1-y)^2} dy,$$

$$x-a = \frac{(a-b)y}{1-y}, \quad x-b = \frac{a-b}{1-y}.$$

代入得

$$\begin{aligned} & \int R \left[x, (x-a)^{\frac{p}{n}} (x-b)^{\frac{q}{n}} \right] dx \\ &= (a-b) \int R \left[\frac{a-by}{1-y}, y^{\frac{p}{n}} \left(\frac{a-b}{1-y} \right)^k \right] \frac{dy}{(1-y)^2}. \end{aligned}$$

再设 $\sqrt[n]{y} = t$, 则 $y = t^n$, $dy = nt^{n-1} dt$, 从而上述积分化为

$$\begin{aligned} & \int R \left[x, (x-a)^{\frac{p}{n}} (x-b)^{\frac{q}{n}} \right] dx \\ &= n(a-b) \int R \left[\frac{a-bt^n}{1-t^n}, t^k \left(\frac{a-b}{1-t^n} \right)^k \right] \frac{t^{n-1}}{(1-t^n)^2} dt, \end{aligned}$$

因为被积函数为 t 的有理函数, 所以积分是初等函数.
求最简单二次无理式的积分:

1937. $\int \frac{x^2}{\sqrt{1+x+x^2}} dx.$

$$\begin{aligned} \text{解} \quad & \int \frac{x^2}{\sqrt{1+x+x^2}} dx = \int \frac{x^2+x+1}{\sqrt{x^2+x+1}} dx \\ & - \frac{1}{2} \int \frac{2x+1}{\sqrt{1+x+x^2}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1+x+x^2}} \\ & = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x+\frac{1}{2}\right) - \frac{1}{2} \int \frac{d(1+x+x^2)}{\sqrt{1+x+x^2}} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\
&= \frac{2x+1}{4} \sqrt{1+x+x^2} + \frac{3}{8} \ln\left(x + \frac{1}{2} + \sqrt{1+x+x^2}\right) \\
& - \sqrt{1+x+x^2} - \frac{1}{2} \ln\left(x + \frac{1}{2} + \sqrt{1+x+x^2}\right) + C \\
&= \frac{2x-3}{4} \sqrt{1+x+x^2} - \frac{1}{8} \ln\left(x + \frac{1}{2} \right. \\
& \quad \left. + \sqrt{1+x+x^2}\right) + C.
\end{aligned}$$

1938⁺. $\int \frac{dx}{(1+x)\sqrt{x^2+x+1}}$.

解 设 $x+1 = \frac{1}{t}$, 则

$$x = \frac{1-t}{t}, \quad dx = -\frac{1}{t^2} dt,$$

$$\sqrt{x^2+x+1} = \frac{\sqrt{t^2-t+1}}{|t|} = \operatorname{sgnt} \cdot \frac{\sqrt{t^2-t+1}}{t}.$$

代入得

$$\begin{aligned}
& \int \frac{dx}{(1+x)\sqrt{x^2+x+1}} = -\operatorname{sgnt} \int \frac{dt}{\sqrt{t^2-t+1}} \\
&= -\operatorname{sgnt} \cdot \ln \left| t - \frac{1}{2} + \sqrt{t^2-t+1} \right| + C_1
\end{aligned}$$

$$= -\operatorname{sgn}(x+1) \cdot \ln \left| \frac{1-x+2[\operatorname{sgn}(x+1)] \cdot \sqrt{x^2+x+1}}{2(x+1)} \right| + C_{10}$$

当 $x+1 > 0$ 时,

$$\int \frac{dx}{(1+x)\sqrt{x^2+x+1}}$$

$$= -\ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C_1$$

当 $x+1 < 0$ 时,

$$\int \frac{dx}{(1+x)\sqrt{x^2+x+1}}$$

$$= \ln \left| \frac{1-x-2\sqrt{x^2+x+1}}{2(1+x)} \right| + C_1$$

$$= \ln \left| \frac{-3(x+1)}{2(1-x+2\sqrt{x^2+x+1})} \right| + C_1$$

$$= -\ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C_1$$

总之,

$$\int \frac{dx}{(1+x)\sqrt{x^2+x+1}}$$

$$= -\ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C_1$$

1939. $\int \frac{dx}{(1-x)^2 \sqrt{1-x^2}}$.

解 设 $\sqrt{\frac{1-x}{1+x}} = t$, 则

$$x = \frac{1-t^2}{1+t^2}, \quad dx = -\frac{4t}{(1+t^2)^2} dt,$$

$$1-x = \frac{2t^2}{1+t^2}, \quad \sqrt{1-x^2} = \frac{2t}{1+t^2}.$$

代入得

$$\int \frac{dx}{(1-x)^2 \sqrt{1-x^2}} = -\frac{1}{2} \int \frac{1+t^2}{t^4} dt$$

$$= \frac{1}{6t^3} + \frac{1}{2t} + C$$

$$= \frac{2-x}{3(1-x)^2} \sqrt{1-x^2} + C.$$

1940. $\int \frac{\sqrt{x^2+2x+2}}{x} dx.$

解 设 $\sqrt{x^2+2x+2} = t-x$, 则

$$x = \frac{t^2-2}{2(t+1)}, \quad dx = \frac{t^2+2t+2}{2(t+1)^2} dt,$$

$$\sqrt{x^2+2x+2} = \frac{t^2+2t+2}{2(t+1)}.$$

代入得

$$\int \frac{\sqrt{x^2+2x+2}}{x} dx = \frac{1}{2} \int \frac{(t^2+2t+2)^2}{(t^2-2)(t+1)^2} dt$$

$$= \frac{1}{2} \int \left[1 + \frac{2}{t+1} - \frac{1}{(t+1)^2} \right] dt$$

$$\begin{aligned}
& - \frac{2\sqrt{2}}{t+\sqrt{2}} + \frac{2\sqrt{2}}{t-\sqrt{2}} \Big] dt \\
& = \frac{t}{2} + \ln|t+1| + \frac{1}{2(t+1)} - \sqrt{2} \ln \left| \frac{t+\sqrt{2}}{t-\sqrt{2}} \right| + C_1 \\
& = \sqrt{x^2+2x+2} + \ln(x+1+\sqrt{x^2+2x+2}) \\
& \quad - \sqrt{2} \ln \left| \frac{x+2+\sqrt{2(x^2+2x+2)}}{x} \right| + C.
\end{aligned}$$

1941. $\int \frac{x dx}{(1+x)\sqrt{1-x-x^2}}$

解 设 $1+x = \frac{1}{t}$, 则

$$x = \frac{1-t}{t}, \quad dx = -\frac{1}{t^2} dt,$$

$$\sqrt{1-x-x^2} = \frac{\sqrt{t^2+t-1}}{|t|} = \operatorname{sgn} t \frac{\sqrt{t^2+t-1}}{t}.$$

代入得

$$\begin{aligned}
& \int \frac{x dx}{(1+x)\sqrt{1-x-x^2}} \\
& = \int \left(\frac{1}{\sqrt{1-x-x^2}} - \frac{1}{(1-x)\sqrt{1-x-x^2}} \right) dx \\
& = \int \frac{dx}{\sqrt{1-x-x^2}} + \operatorname{sgn} t \int \frac{dt}{\sqrt{t^2+t-1}} \\
& = \arcsin \left(\frac{2x+1}{\sqrt{5}} \right) + [\operatorname{sgn}(1+x)]
\end{aligned}$$

$$\cdot \ln \left| \frac{3+x+2(\operatorname{sgn}(x+1))\sqrt{1-x-x^2}}{2(1+x)} \right| + C_1.$$

当 $x+1 > 0$ 时,

$$\int \frac{x dx}{(1+x)\sqrt{1-x-x^2}} = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) \\ + \ln \left| \frac{3+x+2\sqrt{1-x-x^2}}{1+x} \right| + C;$$

当 $x+1 < 0$ 时,

$$\int \frac{x dx}{(1+x)\sqrt{1-x-x^2}} = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) \\ - \ln \left| \frac{3+x-2\sqrt{1-x-x^2}}{2(1+x)} \right| + C_1 \\ = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) + \ln \left| \frac{3+x+2\sqrt{1-x-x^2}}{1+x} \right| + C.$$

总之,

$$\int \frac{x dx}{(1+x)\sqrt{1-x-x^2}} = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) \\ + \ln \left| \frac{3+x+2\sqrt{1-x-x^2}}{1+x} \right| + C.$$

以后诸题中, 出现二次无理式时也会碰到用 *sgn* 的问题, 可参照1938题及1941题类似地处理. 在解这类习题时, 不妨就开方后取正值求解. 如无特殊情况, 今后不再另加说明.

1942. $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}} dx.$

解 $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}} dx = \int \frac{(x^2-x-1)+2}{\sqrt{1+x-x^2}} dx$

$$= -\int \sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2} d\left(x - \frac{1}{2}\right)$$

$$+ 2 \int \frac{d\left(x - \frac{1}{2}\right)}{\sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2}}$$

$$= \frac{1-2x}{4} \sqrt{1+x-x^2} - \frac{5}{8} \arcsin\left(\frac{2x-1}{\sqrt{5}}\right)$$

$$+ 2 \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C$$

$$= \frac{1-2x}{4} \sqrt{1+x-x^2} - \frac{11}{8} \arcsin\left(\frac{1-2x}{\sqrt{5}}\right) + C.$$

利用公式

$$\int \frac{P_n(x)}{y} dx = Q_{n-1}(x)y + \lambda \int \frac{dx}{y},$$

式中 $y = \sqrt{ax^2 + bx + c}$, $P_n(x)$ 为 n 次多项式, $Q_{n-1}(x)$ 为 $n-1$ 次多项式及 λ 为常数, 求下列积分:

1943. $\int \frac{x^3}{\sqrt{1+2x-x^2}} dx.$

解 设 $\int \frac{x^3}{\sqrt{1+2x-x^2}} dx$

$$= (ax^2 + bx + c)\sqrt{1+2x-x^2} + \lambda \int \frac{dx}{\sqrt{1+2x-x^2}},$$

两边对 x 求导数, 得

$$\begin{aligned} \frac{x^3}{\sqrt{1+2x-x^2}} &= (2ax+b)\sqrt{1+2x-x^2} \\ &+ \frac{(ax^2+bx+c)(1-x)}{\sqrt{1+2x-x^2}} + \frac{\lambda}{\sqrt{1+2x-x^2}}. \end{aligned}$$

从而有

$$\begin{aligned} x^3 &\equiv (2ax+b)(1+2x-x^2) + (ax^2+bx+c) \\ &\cdot (1-x) + \lambda. \end{aligned}$$

比较等式两端 x 的同次幂系数, 得

$$\begin{cases} x^3 & -3a=1, \\ x^2 & 5a-2b=0, \\ x^1 & 2a+3b-c=0, \\ x^0 & b+c+\lambda=0. \end{cases}$$

由此, $a = -\frac{1}{3}$, $b = -\frac{5}{6}$, $c = -\frac{19}{6}$, $\lambda = 4$.

于是,

$$\begin{aligned} \int \frac{x^3}{\sqrt{1+2x-x^2}} dx &= -\frac{19+5x+2x^2}{6} \sqrt{1+2x-x^2} \\ &+ 4 \int \frac{dx}{\sqrt{1+2x-x^2}} \\ &= -\frac{19+5x+2x^2}{6} \sqrt{1+2x-x^2} \end{aligned}$$

$$+ 4a \operatorname{arcsin}\left(\frac{x-1}{\sqrt{2}}\right) + C.$$

1944. $\int \frac{x^{10}}{\sqrt{1+x^2}} dx,$

解 设 $\int \frac{x^{10}}{\sqrt{1+x^2}} dx = (ax^9 + bx^8 + cx^7 + dx^6 + ex^5$
 $+ fx^4 + gx^3 + hx^2 + lx + m)\sqrt{1+x^2}$
 $+ \lambda \int \frac{dx}{\sqrt{1+x^2}},$

从而有

$$x^{10} \equiv (9ax^8 + 8bx^7 + 7cx^6 + 6dx^5 + 5ex^4 + 4fx^3$$
 $+ 3gx^2 + 2hx + l)(1+x^2)$
 $+ x(ax^9 + bx^8 + cx^7 + dx^6 + ex^5 + fx^4 + gx^3$
 $+ hx^2 + lx + m) + \lambda.$

比较等式两端 x 的同次幂系数, 求得

$$a = \frac{1}{10}, b = 0, c = -\frac{9}{80}, d = 0,$$

$$e = \frac{21}{160}, f = 0, g = -\frac{21}{128}, h = 0,$$

$$l = \frac{63}{256}, m = 0, \lambda = -\frac{63}{256}.$$

于是,

$$\int \frac{x^{10}}{\sqrt{1+x^2}} dx = \left(\frac{63}{256}x - \frac{21}{128}x^3 + \frac{21}{160}x^5 - \frac{9}{80}x^7 \right.$$

$$+\frac{1}{10}x^9)\sqrt{1+x^2}-\frac{63}{256}\ln(x+\sqrt{1+x^2})+C.$$

1945. $\int x^4 \sqrt{a^2-x^2} dx.$

$$\begin{aligned} \text{解} \quad \int x^4 \sqrt{a^2-x^2} dx &= \int \frac{x^4(a^2-x^2)}{\sqrt{a^2-x^2}} dx \\ &= (Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F) \sqrt{a^2-x^2} \\ &\quad + \lambda \int \frac{dx}{\sqrt{a^2-x^2}}, \end{aligned}$$

从而有

$$\begin{aligned} x^4(a^2-x^2) &\equiv (5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx \\ &\quad + E)(a^2-x^2) - x(Ax^5 + Bx^4 + Cx^3 + Dx^2 \\ &\quad + Ex + F) + \lambda. \end{aligned}$$

比较等式两端 x 的同次幂系数, 求得

$$A = \frac{1}{6}, B = 0, C = -\frac{a^2}{24}, D = 0,$$

$$E = -\frac{a^4}{16}, F = 0, \lambda = \frac{a^4}{16}.$$

于是,

$$\begin{aligned} \int x^4 \sqrt{a^2-x^2} dx &= \left(\frac{1}{6}x^5 - \frac{a^2}{24}x^3 - \frac{a^4}{16}x \right) \sqrt{a^2-x^2} \\ &\quad + \frac{a^4}{16} \arcsin \frac{x}{|a|} + C \quad (a \neq 0). \end{aligned}$$

1946. $\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx.$

解 设 $\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx$

$$= (ax^2 + bx + c)\sqrt{x^2 + 4x + 3} + \lambda \int \frac{dx}{\sqrt{x^2 + 4x + 3}},$$

从而有

$$x^3 - 6x^2 + 11x - 6 \equiv (2ax + b)(x^2 + 4x + 3) \\ + (x + 2)(ax^2 + bx + c) + \lambda.$$

比较等式两端 x 的同次幂系数, 求得

$$a = \frac{1}{3}, \quad b = -\frac{14}{3}, \quad c = 37, \quad \lambda = -66.$$

于是,

$$\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx \\ = \left(\frac{1}{3}x^2 - \frac{14}{3}x + 37 \right) \sqrt{x^2 + 4x + 3} \\ - 66 \ln |x + 2 + \sqrt{x^2 + 4x + 3}| + C.$$

1947. $\int \frac{dx}{x^3 \sqrt{x^2 + 1}}$

解 设 $x = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$, 这里碰到二次无理式 $\sqrt{x^2 + 1}$ 需引用 sgnt 的问题, 不妨设

$$\sqrt{x^2 + 1} = \frac{\sqrt{t^2 + 1}}{t} \quad (t > 0).$$

代入得

$$\begin{aligned}
\int \frac{dx}{x^2 \sqrt{x^2+1}} &= - \int \frac{t^2}{\sqrt{t^2+1}} dt \\
&= - \int \frac{(t^2+1)-1}{\sqrt{t^2+1}} dt \\
&= - \int \sqrt{t^2+1} dt + \int \frac{dt}{\sqrt{t^2+1}} \\
&= -\frac{t}{2} \sqrt{t^2+1} - \frac{1}{2} \ln |t + \sqrt{t^2+1}| \\
&\quad + \ln |t + \sqrt{t^2+1}| + C \\
&= -\frac{\sqrt{x^2+1}}{2x^2} + \frac{1}{2} \ln \frac{1 + \sqrt{x^2+1}}{|x|} + C.
\end{aligned}$$

1948⁺. $\int \frac{dx}{x^4 \sqrt{x^2-1}}$.

解 不妨设 $x = \frac{1}{t} > 0$, 则 $dx = -\frac{1}{t^2} dt$. 由 $|x| > 1$ 知必有 $|t| < 1$, 则有

$$\sqrt{x^2-1} = \frac{\sqrt{1-t^2}}{t} \quad (0 < t < 1).$$

代入得

$$\begin{aligned}
\int \frac{dx}{x^4 \sqrt{x^2-1}} &= - \int \frac{t^3}{\sqrt{1-t^2}} dt \\
&= \int \frac{t(1-t^2)-t}{\sqrt{1-t^2}} dt \\
&= \int t\sqrt{1-t^2} dt - \int \frac{t}{\sqrt{1-t^2}} dt
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \int (1-t^2)^{\frac{1}{2}} d(1-t^2) \\
&\quad + \frac{1}{2} \int (1-t^2)^{-\frac{1}{2}} d(1-t^2) \\
&= -\frac{1}{\frac{3}{2}} (1-t^2)^{\frac{3}{2}} + (1-t^2)^{\frac{1}{2}} + C \\
&= \frac{1+2x^2}{3x^3} \sqrt{x^2-1} + C.
\end{aligned}$$

1949⁺. $\int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}}$.

解 设 $x-1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$. 不妨设 $t > 0$,

则有

$$\sqrt{x^2+3x+1} = \frac{\sqrt{5t^2+5t+1}}{t}.$$

代入得

$$\begin{aligned}
\int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}} &= - \int \frac{t^2}{\sqrt{5t^2+5t+1}} dt \\
&= (at+b) \sqrt{5t^2+5t+1} + \lambda \int \frac{dt}{\sqrt{5t^2+5t+1}},
\end{aligned}$$

从而

$$-t^2 \equiv a(1+5t+5t^2) + \left(5t + \frac{5}{2}\right)(at+b) + \lambda.$$

比较等式两端 t 的同次幂系数, 求得

$$a = -\frac{1}{10}, \quad b = \frac{3}{20}, \quad \lambda = -\frac{11}{40}.$$

于是,

$$\begin{aligned} \int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}} &= \left(-\frac{t}{10} + \frac{3}{20}\right) \sqrt{5t^2+5t+1} \\ &\quad - \frac{11}{40} \int \frac{dt}{\sqrt{5t^2+5t+1}} \\ &= \frac{3-2t}{20} \sqrt{5t^2+5t+1} - \frac{11}{40\sqrt{5}} \ln \left| t + \frac{1}{2} \right. \\ &\quad \left. + \sqrt{t^2+t+\frac{1}{5}} \right| + C_1 \\ &= \frac{3x-5}{20(x-1)^2} \sqrt{x^2+3x+1} \\ &\quad - \frac{11}{40\sqrt{5}} \ln \left| \frac{\sqrt{5}(x+1)+2\sqrt{x^2+3x+1}}{x-1} \right| + C. \end{aligned}$$

$$1950^+. \int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}}.$$

解 设 $x+1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$. 先设 $t > 0$, 则有

$$\sqrt{x^2+2x} = \frac{\sqrt{1-t^2}}{t}.$$

代入得

$$\int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}} = - \int \frac{t^4}{\sqrt{1-t^2}} dt$$

$$= (at^3 + bt^2 + ct + e)\sqrt{1-t^2} + \lambda \int \frac{dt}{\sqrt{1-t^2}}.$$

从而有

$$-t^4 \equiv (3at^2 + 2bt + c)(1-t^2) - t(at^3 + bt^2 + ct + e) + \lambda.$$

比较等式两端 t 的同次幂系数, 求得

$$a = \frac{1}{4}, \quad b = 0, \quad c = \frac{3}{8}, \quad e = 0, \quad \lambda = -\frac{3}{8}.$$

于是,

$$\begin{aligned} \int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}} &= \left(\frac{1}{4}t^3 + \frac{3}{8}t \right) \\ &\cdot \sqrt{1-t^2} - \frac{3}{8} \int \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{3x^2+6x+5}{8(x+1)^4} \sqrt{x^2+2x} - \frac{3}{8} \arcsin \frac{1}{x+1} + C. \end{aligned}$$

再设 $t < 0$, 则答案前一项不改变符号, 但后一项要改变符号, 因此, 最后得到

$$\begin{aligned} \int \frac{dx}{(x+1)^5 \sqrt{x^2+2x}} &= \frac{3x^2+6x+5}{8(x+1)^4} \sqrt{x^2+2x} \\ &- \frac{3}{8} \arcsin \frac{1}{|x+1|} + C, \end{aligned}$$

其中 $x > 0$ 或 $x < -2$.

1951. 在什么条件下, 积分

$$\int \frac{a_1 x^2 + b_1 x + c_1}{\sqrt{ax^2 + bx + c}} dx$$

是代数函数?

解 设 $\int \frac{a_1x^2 + b_1x + c_1}{\sqrt{ax^2 + bx + c}} dx$

$$= (Ax + B)\sqrt{ax^2 + bx + c} + \lambda \int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

从而有

$$a_1x^2 + b_1x + c_1 \equiv A(ax^2 + bx + c)$$

$$+ \left(ax + \frac{b}{2}\right)(Ax + B) + \lambda.$$

比较等式两端 x 的同次幂系数, 当 $a \neq 0$ 时求得

$$A = \frac{a_1}{2a}, \quad B = \frac{4ab_1 - 3a_1b}{4a^2},$$

$$\lambda = \frac{8a^2c_1 + 3a_1b^2 - 4a(a_1c + bb_1)}{8a^2}.$$

于是, 当 $a \neq 0$ 且 $8a^2c_1 + 3a_1b^2 = 4a(a_1c + bb_1)$ 时, $\lambda = 0$, 积分为代数函数; 当 $a = 0$ 时积分显然为代数函数.

要求积分 $\int \frac{P(x)}{Q(x)y} dx$, 式中 $y = \sqrt{ax^2 + bx + c}$,

应先分解有理函数 $\frac{P(x)}{Q(x)}$ 为最简分式.

1952. $\int \frac{x dx}{(x-1)^2 \sqrt{1+2x-x^2}}$

解 $\int \frac{x dx}{(x-1)^2 \sqrt{1+2x-x^2}} = \int \frac{dx}{(x-1)^2 \sqrt{1+2x-x^2}}$

$$+ \int \frac{dx}{(x-1)\sqrt{1+2x-x^2}}.$$

设 $x-1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2} dt$. 不妨设 $t > 0$,

则有

$$\sqrt{1+2x-x^2} = \frac{\sqrt{2t^2-1}}{t}.$$

代入得

$$\begin{aligned} & \int \frac{x dx}{(x-1)^2 \sqrt{1+2x-x^2}} \\ &= - \int \frac{t dt}{\sqrt{2t^2-1}} - \int \frac{dt}{\sqrt{2t^2-1}} \\ &= -\frac{1}{2} \sqrt{2t^2-1} - \frac{1}{\sqrt{2}} \ln |\sqrt{2}t + \sqrt{2t^2-1}| + C \\ &= \frac{\sqrt{1+2x-x^2}}{2(1-x)} - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sqrt{1+2x-x^2}}{1-x} \right| + C. \end{aligned}$$

1953. $\int \frac{x dx}{(x^2-1)\sqrt{x^2-x-1}}.$

解 $\int \frac{x dx}{(x^2-1)\sqrt{x^2-x-1}}$

$$= \frac{1}{2} \int \left(\frac{1}{x+1} + \frac{1}{x-1} \right) \frac{dx}{\sqrt{x^2-x-1}}$$

$$= \frac{1}{2} \int \frac{dx}{(x+1)\sqrt{x^2-x-1}} + \frac{1}{2} \int \frac{dx}{(x-1)\sqrt{x^2-x-1}}$$

$$= \frac{1}{2}I_1 + \frac{1}{2}I_2.$$

对于 I_1 , 设 $x+1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2}dt$. 不妨设 $t > 0$, 则有

$$\sqrt{x^2 - x - 1} = \frac{\sqrt{t^2 - 3t + 1}}{t}.$$

代入 I_1 , 得

$$\begin{aligned} I_1 &= \int \frac{dx}{(x+1)\sqrt{x^2-x-1}} = - \int \frac{dt}{\sqrt{t^2-3t+1}} \\ &= - \ln \left| t - \frac{3}{2} + \sqrt{t^2-3t+1} \right| + C_1 \\ &= - \ln \left| \frac{3x+1-2\sqrt{x^2-x-1}}{x+1} \right| + C_2 \end{aligned}$$

对于 I_2 , 设 $x-1 = \frac{1}{t}$, 同上可得

$$\begin{aligned} I_2 &= \int \frac{dx}{(x-1)\sqrt{x^2-x-1}} \\ &= \arcsin \left(\frac{x-3}{|x-1|\sqrt{5}} \right) + C_3. \end{aligned}$$

于是,

$$\begin{aligned} &\int \frac{dx}{(x^2-1)\sqrt{x^2-x-1}} \\ &= -\frac{1}{2} \ln \left| \frac{3x+1-2\sqrt{x^2-x-1}}{x+1} \right| \end{aligned}$$

$$+\frac{1}{2}\arcsin\left(\frac{x-3}{|x-1|\sqrt{5}}\right)+C.$$

1954. $\int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx.$

解
$$\int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx = \int \frac{x^2+x+1}{(x+1)^2} \cdot \frac{dx}{\sqrt{x^2+x+1}}$$

$$= \int \frac{(x+1)^2 - (x+1) + 1}{(x+1)^2} \cdot \frac{dx}{\sqrt{x^2+x+1}}$$

$$= \int \frac{dx}{\sqrt{x^2+x+1}} - \int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$+ \int \frac{dx}{(x+1)^2\sqrt{x^2+x+1}} = I_1 - I_2 + I_3.$$

对于 I_1 , 显然有

$$I_1 = \int \frac{dx}{\sqrt{x^2+x+1}} = \ln\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right) + C_1;$$

对于 I_2 , 利用1938题的结果, 即得

$$I_2 = \int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= -\ln\left|\frac{1-x+2\sqrt{x^2+x+1}}{x+1}\right| + C_2;$$

对于 I_3 , 设 $x+1 = \frac{1}{t}$, 则 $dx = -\frac{1}{t^2}dt$. 不妨设 $t > 0$.

则有

$$\sqrt{x^2+x+1} = \frac{\sqrt{t^2-t+1}}{t}.$$

代入得

$$\begin{aligned} I_3 &= - \int \frac{t dt}{\sqrt{t^2 - t + 1}} \\ &= - \frac{1}{2} \int \frac{(2t-1) dt}{\sqrt{t^2 - t + 1}} - \frac{1}{2} \int \frac{dt}{\sqrt{t^2 - t + 1}} \\ &= - \sqrt{t^2 - t + 1} - \frac{1}{2} \ln \left| t - \frac{1}{2} + \sqrt{t^2 - t + 1} \right| + C_3 \\ &= - \frac{\sqrt{x^2 + x + 1}}{x+1} - \frac{1}{2} \ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| \\ &\quad + C_4. \end{aligned}$$

于是, 最后得到

$$\begin{aligned} &\int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx \\ &= \ln \left(x + \frac{1}{2} + \sqrt{x^2+x+1} \right) - \frac{\sqrt{x^2+x+1}}{x+1} \\ &\quad + \frac{1}{2} \ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right| + C. \end{aligned}$$

如用下述解法更简单些:

$$\begin{aligned} \int \frac{\sqrt{x^2+x+1}}{(x+1)^2} dx &= - \int \sqrt{x^2+x+1} d\left(\frac{1}{x+1}\right) \\ &= - \frac{\sqrt{x^2+x+1}}{x+1} + \int \frac{\left(x+\frac{1}{2}\right) dx}{(x+1)\sqrt{x^2+x+1}} \end{aligned}$$

$$\begin{aligned}
&= -\frac{\sqrt{x^2+x+1}}{x+1} + \int \frac{dx}{\sqrt{x^2+x+1}} \\
&\quad - \frac{1}{2} \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} \\
&= -\frac{\sqrt{x^2+x+1}}{x+1} + \ln\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right) \\
&\quad + \frac{1}{2} \ln \left| \frac{1-x+2\sqrt{x^2+x+1}}{x+1} \right|^{(*)} + C.
\end{aligned}$$

*) 利用1938题的结果。

1955. $\int \frac{x^3}{(1+x)\sqrt{1+2x-x^2}} dx.$

$$\begin{aligned}
\text{解} \quad & \int \frac{x^3}{(1+x)\sqrt{1+2x-x^2}} dx \\
&= \int \frac{(x^2+1)-1}{(1+x)\sqrt{1+2x-x^2}} dx \\
&= \int \frac{x^2-x+1}{\sqrt{1+2x-x^2}} dx - \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} \\
&= -\int \frac{1+2x-x^2}{\sqrt{1+2x-x^2}} dx + \frac{1}{2} \int \frac{(2x-2)dx}{\sqrt{1+2x-x^2}} \\
&\quad + 3 \int \frac{dx}{\sqrt{1+2x-x^2}} - \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}} \\
&= -\int \sqrt{2-(x-1)^2} d(x-1) - \frac{1}{2} \int \frac{d(1+2x-x^2)}{\sqrt{1+2x-x^2}} \\
&\quad + 3 \int \frac{d(x-1)}{\sqrt{2-(x-1)^2}} - I_1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1-x}{2} \sqrt{1+2x-x^2} - \arcsin\left(\frac{x-1}{\sqrt{2}}\right) \\
&\quad - \sqrt{1+2x-x^2} + 3 \arcsin\left(\frac{x-1}{\sqrt{2}}\right) - I_1 \\
&= -\frac{x+1}{2} \sqrt{1+2x-x^2} + 2 \arcsin\left(\frac{x-1}{\sqrt{2}}\right) - I_1.
\end{aligned}$$

对于 I_1 , 设 $x+1 = \frac{1}{t}$, 可得

$$\begin{aligned}
I_1 &= \int \frac{dx}{(x+1)\sqrt{1+2x-x^2}} \\
&= \frac{1}{\sqrt{2}} \arcsin\left(\frac{x\sqrt{2}}{x+1}\right) + C_1.
\end{aligned}$$

于是, 最后得到

$$\begin{aligned}
&\int \frac{x^3}{(x+1)\sqrt{1+2x-x^2}} dx \\
&= -\frac{1+x}{2} \sqrt{1+2x-x^2} - 2 \arcsin\left(\frac{1-x}{\sqrt{2}}\right) \\
&\quad - \frac{1}{\sqrt{2}} \arcsin\left(\frac{x\sqrt{2}}{1+x}\right) + C.
\end{aligned}$$

1956. $\int \frac{xdx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$

解 $\int \frac{xdx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$

$$= \int \left(\frac{2}{x-2} - \frac{1}{x-1}\right) \cdot \frac{dx}{\sqrt{x^2-4x+3}}$$

$$= \int \frac{2dx}{(x-2)\sqrt{x^2-4x+3}} - \int \frac{dx}{(x-1)\sqrt{x^2-4x+3}}$$

$$= 2I_1 - I_2.$$

对于 I_1 , 设 $x-2 = \frac{1}{t}$, 可得

$$I_1 = \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$$

$$= -\arcsin\left(\frac{1}{|x-2|}\right) + C_1;$$

对于 I_2 , 设 $x-1 = \frac{1}{t}$, 可得

$$I_2 = \int \frac{dx}{(x-1)\sqrt{x^2-4x+3}} = \frac{\sqrt{x^2-4x+3}}{x-1} + C_2.$$

于是, 最后得到

$$\int \frac{x dx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$$

$$= -2\arcsin\left(\frac{1}{|x-2|}\right) - \frac{\sqrt{x^2-4x+3}}{x-1} + C,$$

其中 $x < 1$ 或 $x > 3$.

1957. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}.$

解 设 $x = \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$dx = \cos t dt, \quad \sqrt{1-x^2} = \cos t.$$

代入得

$$\begin{aligned}
& \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{dt}{1+\sin^2 t} \\
& = \int \frac{dt}{2\sin^2 t + \cos^2 t} = \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2} \operatorname{tg} t)}{(\sqrt{2} \operatorname{tg} t)^2 + 1} \\
& = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg}(\sqrt{2} \operatorname{tg} t) + C \\
& = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{x\sqrt{2}}{\sqrt{1-x^2}} \right) + C.
\end{aligned}$$

1958. $\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}$.

解 当 $x > 1$ 时, 设 $x = \operatorname{sect}$, 并限制 $0 < t < \frac{\pi}{2}$, 则

$$dx = \operatorname{sect} \cdot \operatorname{tg} t dt, \quad \sqrt{x^2-1} = \operatorname{tg} t.$$

代入得

$$\begin{aligned}
& \int \frac{dx}{(x^2+1)\sqrt{x^2-1}} = \int \frac{\operatorname{sect} dt}{1+\sec^2 t} \\
& = \int \frac{\cos t}{\cos^2 t + 1} dt = \int \frac{d(\sin t)}{2-\sin^2 t} \\
& = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sin t}{\sqrt{2} - \sin t} \right| + C \\
& = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}x + \sqrt{x^2-1}}{\sqrt{2}x - \sqrt{x^2-1}} \right| + C.
\end{aligned}$$

当 $x < -1$ 时, 仍设 $x = \operatorname{sect}$, 但限制 $\pi < t < \frac{3}{2}\pi$,

经计算可获得同样的结果.

总之, 当 $|x| > 1$ 时,

$$\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{x\sqrt{2} + \sqrt{x^2-1}}{x\sqrt{2} - \sqrt{x^2-1}} \right| + C.$$

1959. $\int \frac{dx}{(1-x^4)\sqrt{1+x^2}}.$

解 设 $x = \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$ 且 $|t| \neq \frac{\pi}{4}$, 则

$$dx = \sec^2 t dt, \quad \sqrt{1+x^2} = \sec t.$$

代入得

$$\int \frac{dx}{(1-x^4)\sqrt{x^2+1}} = \int \frac{\sec^2 t dt}{(1-\operatorname{tg}^4 t)\sec t}$$

$$= \int \frac{\cos^3 t dt}{1-2\sin^2 t} = \int \frac{1-\sin^2 t}{1-2\sin^2 t} d(\sin t)$$

$$= \frac{1}{2} \int \frac{1-2\sin^2 t}{1-2\sin^2 t} d(\sin t) + \frac{1}{2} \int \frac{d(\sin t)}{1-2\sin^2 t}$$

$$= \frac{1}{2} \sin t + \frac{1}{4\sqrt{2}} \ln \left| \frac{1+\sqrt{2}\sin t}{1-\sqrt{2}\sin t} \right| + C$$

$$= \frac{x}{2\sqrt{1+x^2}} + \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{1+x^2} + x\sqrt{2}}{\sqrt{1+x^2} - x\sqrt{2}} \right|$$

$$+ C \quad (|x| \neq 1).$$

1960. $\int \frac{\sqrt{x^2+2}}{x^2+1} dx.$

$$\begin{aligned}
 \text{解} \quad \int \frac{\sqrt{x^2+2}}{x^2+1} dx &= \int \frac{(x^2+2)dx}{(x^2+1)\sqrt{x^2+2}} \\
 &= \int \left(1 + \frac{1}{x^2+1}\right) \cdot \frac{dx}{\sqrt{x^2+2}} \\
 &= \int \frac{dx}{\sqrt{x^2+2}} + \int \frac{dx}{(x^2+1)\sqrt{x^2+2}} \\
 &= \ln(x + \sqrt{x^2+2}) + I_1.
 \end{aligned}$$

对于 I_1 , 设 $x = \sqrt{2} \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$dx = \sqrt{2} \sec^2 t dt, \quad \sqrt{x^2+2} = \sqrt{2} \sec t.$$

代入得

$$\begin{aligned}
 I_1 &= \int \frac{dx}{(x^2+1)\sqrt{x^2+2}} = \int \frac{\sec t dt}{1+2\operatorname{tg}^2 t} \\
 &= \int \frac{\cos t dt}{1+\sin^2 t} = \int \frac{d(\sin t)}{1+\sin^2 t} = \operatorname{arctg}(\sin t) + C_1 \\
 &= \operatorname{arctg} \left(\frac{x}{\sqrt{2+x^2}} \right) + C_1 \\
 &= -\operatorname{arctg} \left(\frac{\sqrt{x^2+2}}{x} \right) + C.
 \end{aligned}$$

于是, 最后得到

$$\begin{aligned}
 \int \frac{\sqrt{x^2+2}}{x^2+1} dx &= \ln(x + \sqrt{x^2+2}) \\
 &\quad - \operatorname{arctg} \left(\frac{\sqrt{x^2+2}}{x} \right) + C.
 \end{aligned}$$

化二次三项式为正则型，以计算下列积分：

$$1961. \int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}}.$$

$$\text{解} \int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}}$$

$$= \int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right]\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{5}{4}}}.$$

当 $x+\frac{1}{2} > \frac{\sqrt{5}}{2}$ 时，设 $x+\frac{1}{2} = \frac{\sqrt{5}}{2} \sec t$ ，并限

制 $0 < t < \frac{\pi}{2}$ ，则

$$dx = \frac{\sqrt{5}}{2} \sec t \cdot \operatorname{tg} t dt, \quad \sqrt{x^2+x-1} = \frac{\sqrt{5}}{2} \operatorname{tg} t,$$

$$x^2+x+1 = \frac{1}{4}(5\sec^2 t + 3).$$

代入得

$$\begin{aligned} & \int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}} \\ &= 4 \int \frac{\sec t dt}{5\sec^2 t + 3} = 4 \int \frac{\cos t dt}{5 + 3\cos^2 t} \\ &= 4 \cdot \frac{1}{\sqrt{3}} \int \frac{d(\sqrt{3} \sin t)}{(\sqrt{8})^2 - (\sqrt{3} \sin t)^2} \\ &= 4 \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{2\sqrt{8}} \ln \left| \frac{\sqrt{8} + \sqrt{3} \sin t}{\sqrt{8} - \sqrt{3} \sin t} \right| + C \end{aligned}$$

$$= \frac{1}{\sqrt{6}} \ln \left| \frac{(2x+1)\sqrt{2} + \sqrt{3(x^2+x-1)}}{(2x+1)\sqrt{2} - \sqrt{3(x^2+x-1)}} \right| + C.$$

当 $x + \frac{1}{2} < -\frac{\sqrt{5}}{2}$ 时, 仍设 $x + \frac{1}{2} = \frac{\sqrt{5}}{2} \operatorname{sect}$,

但限制 $\pi < t < \frac{3}{2}\pi$. 经计算可获同样的结果.

总之, 当 $|x + \frac{1}{2}| > \frac{\sqrt{5}}{2}$ 时,

$$\begin{aligned} & \int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}} \\ &= \frac{1}{\sqrt{6}} \ln \left| \frac{(2x+1)\sqrt{2} + \sqrt{3(x^2+x-1)}}{(2x+1)\sqrt{2} - \sqrt{3(x^2+x-1)}} \right| + C. \end{aligned}$$

1962. $\int \frac{x^2 dx}{(4-2x+x^2)\sqrt{2+2x-x^2}}.$

解 $\int \frac{x^2 dx}{(4-2x+x^2)\sqrt{2+2x-x^2}}$

$$= \int \frac{(x-1)^2 + 2(x-1) + 1}{[3+(x-1)^2]\sqrt{3-(x-1)^2}} dx.$$

设 $x-1 = \sqrt{3} \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$dx = \sqrt{3} \cos t dt, \quad \sqrt{2+2x-x^2} = \sqrt{3} \cos t.$$

代入得

$$\begin{aligned} & \int \frac{x^2 dx}{(4-2x+x^2)\sqrt{2+2x-x^2}} \\ &= \int \frac{1 + 2\sqrt{3} \sin t + 3 \sin^2 t}{3(1 + \sin^2 t)} dt \end{aligned}$$

$$= \int dt + \frac{2}{\sqrt{3}} \int \frac{\sin t}{1 + \sin^2 t} dt - \frac{2}{3} \int \frac{dt}{1 + \sin^2 t}$$

$$= t - \frac{2}{\sqrt{3}} \int \frac{d(\cos t)}{2 - \cos^2 t} - \frac{2}{3} \int \frac{d(\operatorname{tg} t)}{1 + 2\operatorname{tg}^2 t}$$

$$= t - \frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{2} + \cos t}{\sqrt{2} - \cos t} \right|$$

$$= \frac{\sqrt{2}}{3} \operatorname{arc} \operatorname{tg}(\sqrt{2} \operatorname{tg} t) + C$$

$$= \operatorname{arc} \sin \frac{x-1}{\sqrt{3}} - \frac{1}{\sqrt{6}} \ln \frac{\sqrt{6} + \sqrt{2+2x-x^2}}{\sqrt{6} - \sqrt{2+2x-x^2}}$$

$$= \frac{\sqrt{2}}{3} \operatorname{arc} \operatorname{tg} \frac{(x-1)\sqrt{2}}{\sqrt{2+2x-x^2}} + C.$$

1963. $\int \frac{(x+1)dx}{(x^2+x+1)\sqrt{x^2+x+1}}$

解 $\int \frac{(x+1)dx}{(x^2+x+1)\sqrt{x^2+x+1}}$

$$= \int \frac{\left(x + \frac{1}{2}\right) + \frac{1}{2}}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}} d\left(x + \frac{1}{2}\right)$$

$$= \int \frac{\left(x + \frac{1}{2}\right) d\left(x + \frac{1}{2}\right)}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}} + \frac{1}{2} \int \frac{d\left(x + \frac{1}{2}\right)}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}}$$

$$= \frac{1}{2} \int \frac{d\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^{\frac{3}{2}}} + \frac{1}{2} \cdot \frac{x + \frac{1}{2}}{\frac{3}{4}\sqrt{x^2+x+1}} \quad *)$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2x+1}{3\sqrt{x^2+x+1}} + C$$

$$= \frac{2(x-1)}{3\sqrt{x^2+x+1}} + C.$$

*) 利用1781题的结果。

1964⁺. 利用线性分式的代换 $x = \frac{\alpha + \beta t}{1+t}$, 计算积分:

$$\int \frac{dx}{(x^2-x+1)\sqrt{x^2+x+1}}.$$

解 线性分式的代换

$$x = \frac{\alpha + \beta t}{1+t}$$

给出

$$x^2 \pm x + 1 = \frac{(\beta^2 \pm \beta + 1)t^2 + [2\alpha\beta \pm (\alpha + \beta) + 2]t + (\alpha^2 \pm \alpha + 1)}{(1+t)^2}.$$

要求 $2\alpha\beta \pm (\alpha + \beta) + 2 = 0$ 即化成正则型, 当 $\alpha + \beta = 0$ 及 $\alpha\beta = -1$ 时即得上式。例如, 取

$$\alpha = -1, \beta = 1,$$

我们有

$$x = \frac{t-1}{1+t} \text{ 或 } t = \frac{1+x}{1-x},$$

$$dx = \frac{2dt}{(1+t)^2}, \quad x^2 - x + 1 = \frac{t^2 + 3}{(t+1)^2},$$

$$\sqrt{x^2+x+1} = \frac{\sqrt{1+3t^2}}{t+1},$$

其中不妨设 $t+1 \geq 0$ 。

于是,

$$\begin{aligned} & \int \frac{dx}{(x^2-x+1)\sqrt{x^2+x+1}} \\ &= 2 \int \frac{t+1}{(t^2+3)\sqrt{1+3t^2}} dt \\ &= 2 \int \frac{tdt}{(t^2+3)\sqrt{1+3t^2}} + 2 \int \frac{dt}{(t^2+3)\sqrt{1+3t^2}} \\ &= 2(I_1+I_2). \end{aligned}$$

对于 I_1 , 设 $u = \sqrt{1+3t^2}$, 则

$$du = \frac{3tdt}{\sqrt{1+3t^2}}, \quad t^2+3 = \frac{u^2+8}{3}.$$

代入得

$$\begin{aligned} I_1 &= \int \frac{tdt}{(t^2+3)\sqrt{1+3t^2}} = \int \frac{du}{u^2+8} \\ &= \frac{1}{2\sqrt{2}} \operatorname{arc\,tg} \left(\frac{u}{2\sqrt{2}} \right) + C_1 \\ &= \frac{1}{2\sqrt{2}} \operatorname{arc\,tg} \left[\frac{\sqrt{x^2+x+1}}{(1-x)\sqrt{2}} \right] + C_1. \end{aligned}$$

对于 I_2 , 设 $u = \frac{3t}{\sqrt{1+3t^2}}$, 则

$$\frac{dt}{\sqrt{1+3t^2}} = \frac{du}{3-u^2}, \quad t^2+3 = \frac{27-8u^2}{3(3-u^2)},$$

代入得

$$\begin{aligned}
I_2 &= \int \frac{dt}{(t^2+3)\sqrt{1+3t^2}} = 3 \int \frac{du}{27-8u^2} \\
&= \frac{1}{4\sqrt{6}} \ln \left| \frac{3\sqrt{3} + 2\sqrt{2}u}{3\sqrt{3} - 2\sqrt{2}u} \right| + C_2 \\
&= \frac{1}{4\sqrt{6}} \ln \left| \frac{\sqrt{3(x^2+x+1)} + (x+1)\sqrt{2}}{\sqrt{3(x^2+x+1)} - (x+1)\sqrt{2}} \right| + C_2 \\
&= \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{x^2-x+1}}{\sqrt{3(x^2+x+1)} - (x+1)\sqrt{2}} \right| + C_2.
\end{aligned}$$

于是，最后得到

$$\begin{aligned}
&\int \frac{dx}{(x^2-x+1)\sqrt{x^2+x+1}} \\
&= -\frac{1}{\sqrt{2}} \operatorname{arctg} \left[\frac{\sqrt{x^2+x+1}}{(x-1)\sqrt{2}} \right] \\
&\quad + \frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{x^2-x+1}}{\sqrt{3(x^2+x+1)} - (x+1)\sqrt{2}} \right| + C.
\end{aligned}$$

1965⁺. 求

$$\int \frac{dx}{(x^2+2)\sqrt{2x^2-2x+5}}.$$

解 此题与1964题均属于下述类型的积分

$$\int \frac{Mx+N}{(x^2+px+q)^m \sqrt{ax^2+bx+c}}$$

【参看微积分学教程 (Γ. M. 菲赫金哥尔茨) 第二卷 第一分册55页“272. 其它的计算方法”】

设 $x = \frac{\alpha + \beta t}{1+t}$, 适当选择 α 与 β , 使得在两个三

项式中同时消去一次项。为此，将 $x = \frac{a + \beta t}{1 + t}$ 分别代入 $x^2 + 2$ 及 $2x^2 - 2x + 5$ 中，并令一次项的系数等于零，求得

$$a = -1, \beta = 2,$$

即设

$$x = \frac{2t - 1}{1 + t}.$$

从而有

$$dx = \frac{3}{(t+1)^2} dt, \quad x^2 + 2 = \frac{3(2t^2 + 1)}{(t+1)^2},$$

$$\sqrt{2x^2 - 2x + 5} = \frac{3\sqrt{t^2 + 1}}{|t+1|}.$$

以下不妨设 $t+1 > 0$ ，

代入得

$$\begin{aligned} & \int \frac{dx}{(x^2 + 2)\sqrt{2x^2 - 2x + 5}} \\ &= \frac{1}{3} \int \frac{t+1}{(2t^2 + 1)\sqrt{t^2 + 1}} dt \\ &= \frac{1}{3} \int \frac{t dt}{(2t^2 + 1)\sqrt{t^2 + 1}} + \frac{1}{3} \int \frac{dt}{(2t^2 + 1)\sqrt{t^2 + 1}}. \end{aligned}$$

对于右端的第一个积分，设 $u = \sqrt{t^2 + 1}$ ，代入后

计算得

$$\frac{1}{3} \int \frac{t dt}{(2t^2 + 1)\sqrt{t^2 + 1}} = \frac{1}{3} \int \frac{du}{2u^2 - 1}$$

$$= \frac{1}{6\sqrt{2}} \ln \frac{\sqrt{\frac{2}{3}u-1}}{\sqrt{\frac{2}{3}u+1}} + C_1$$

$$= \frac{1}{6\sqrt{2}} \ln \frac{\sqrt{2(2x^2-2x+5)}+(x-2)}{\sqrt{2(2x^2-2x+5)}-(x-2)} + C_1.$$

对于右端的第二个积分, 设 $u = \frac{t}{\sqrt{t^2+1}}$, 代入后

计算得

$$\frac{1}{3} \int \frac{dt}{(2t^2+1)\sqrt{t^2+1}} = \frac{1}{3} \int \frac{du}{1+u^2}$$

$$= \frac{1}{3} \operatorname{arctg} u + C_2 = \frac{1}{3} \operatorname{arctg} \left(\frac{1+x}{\sqrt{2x^2-2x+5}} \right) + C_2$$

$$= -\frac{1}{3} \operatorname{arctg} \left(\frac{\sqrt{2x^2-2x+5}}{x+1} \right) + C_3.$$

于是, 最后得到

$$\int \frac{dx}{(x^2+2)\sqrt{2x^2-2x+5}}$$

$$= \frac{1}{6\sqrt{2}} \ln \frac{\sqrt{2(2x^2-2x+5)}+(x-2)}{\sqrt{2(2x^2-2x+5)}-(x-2)}$$

$$- \frac{1}{3} \operatorname{arctg} \left(\frac{\sqrt{2x^2-2x+5}}{x+1} \right) + C.$$

利用尤拉代换

(1) 若 $a > 0$, $\sqrt{ax^2+bx+c} = \pm \sqrt{ax+z^2}$

(2) 若 $c > 0$, $\sqrt{ax^2+bx+c} = xz \pm \sqrt{c}$;

(3) $\sqrt{a(x-x_1)(x-x_2)} = z(x-x_1)$.

以求下列积分:

$$1966. \int \frac{dx}{x + \sqrt{x^2 + x + 1}}.$$

解 设 $\sqrt{x^2 + x + 1} = z - x$, 则

$$x = \frac{z^2 - 1}{1 + 2z}, \quad dx = \frac{2(z^2 + z + 1)}{(1 + 2z)^2} dz,$$

$$\sqrt{x^2 + x + 1} = \frac{z^2 + z + 1}{1 + 2z}.$$

代入得

$$\begin{aligned} \int \frac{dx}{x + \sqrt{x^2 + x + 1}} &= \frac{1}{2} \int \frac{z^2 + z + 1}{z \left(z + \frac{1}{2}\right)^2} dz \\ &= \frac{1}{2} \int \left[\frac{4}{z} - \frac{3}{z + \frac{1}{2}} - \frac{3}{2 \left(z + \frac{1}{2}\right)^2} \right] dz \\ &= \frac{1}{2} \ln \left| \frac{z^4}{z + \frac{1}{2}} \right|^3 + \frac{3}{4 \left(z + \frac{1}{2}\right)} + C_1 \\ &= \frac{1}{2} \ln \left| \frac{z^4}{2z + 1} \right|^3 + \frac{3}{2(2z + 1)} + C, \end{aligned}$$

其中 $z = x + \sqrt{x^2 + x + 1}$.

$$1967. \int \frac{dx}{1 + \sqrt{1 - 2x - x^2}}.$$

解 设 $\sqrt{1 - 2x - x^2} = xz - 1$, 则

$$z = \frac{1 + \sqrt{1 - 2x - x^2}}{x}, \quad x = \frac{2(z - 1)}{z^2 + 1}.$$

$$dx = \frac{2(1+2z-z^2)}{(z^2+1)^2} dz,$$

$$\sqrt{1-2x-x^2} + 1 = \frac{2z(z-1)}{z^2+1}.$$

代入得

$$\begin{aligned} \int \frac{dx}{1+\sqrt{1-2x-x^2}} &= \int \frac{1+2z-z^2}{z(z-1)(z^2+1)} dz \\ &= \int \left[\frac{1}{z-1} - \frac{1}{z} - \frac{2}{z^2+1} \right] dz \\ &= \ln \left| \frac{z-1}{z} \right| - 2\operatorname{arctg} z + C, \end{aligned}$$

其中 $z = \frac{1 + \sqrt{1-2x-x^2}}{x}$.

1968. $\int x\sqrt{x^2-2x+2} dx.$

解 设 $\sqrt{x^2-2x+2} = z-x$, 则

$$x = \frac{z^2-2}{2(z-1)}, \quad dx = \frac{z^2-2z+2}{2(z-1)^2} dz,$$

$$\sqrt{x^2-2x+2} = \frac{z^2-2z+2}{2(z-1)}.$$

代入得

$$\begin{aligned} &\int x\sqrt{x^2-2x+2} dx \\ &= \frac{1}{8} \int \frac{(z^2-2)(z^2-2z+2)^2}{(z-1)^4} dz \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \int \frac{[(z-1)^2 + 2(z-1) - 1] \cdot [(z-1)^2 + 1]^2}{(z-1)^4} dz \\
&= \frac{1}{8} \int \left\{ \left[(z-1)^2 - (z-1)^{-4} \right] \right. \\
&\quad + \left[2(z-1) + 2(z-1)^{-3} \right] \\
&\quad + \left. \left[1 - (z-1)^{-2} \right] + 4(z-1)^{-1} \right\} d(z-1) \\
&= \frac{1}{8} \left\{ \frac{1}{3} \left[(z-1)^3 + (z-1)^{-3} \right] \right. \\
&\quad + \left[(z-1)^2 - (z-1)^{-2} \right] \\
&\quad + \left. \left[(z-1) + (z-1)^{-1} \right] \right\} + \frac{1}{2} \ln|z-1| + C,
\end{aligned}$$

其中 $z = x + \sqrt{x^2 - 2x + 2}$.

1969. $\int \frac{x - \sqrt{x^2 - 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx.$

解 设 $\sqrt{x^2 + 3x + 2} = z(x+1)$, 则

$$x = \frac{2 - z^2}{z^2 - 1}, dx = -\frac{2z}{(z^2 - 1)^2} dz,$$

$$\sqrt{x^2 + 3x + 2} = \frac{z}{z^2 - 1}.$$

代入得

$$\int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx$$

$$\begin{aligned}
&= \int \frac{2z(2-z-z^2)}{(z^2-z-2)(z^2-1)^2} dz \\
&= \int \left[-\frac{17}{108(z+1)} + \frac{5}{18(z+1)^2} + \frac{1}{3(z+1)^3} \right. \\
&\quad \left. + \frac{3}{4(z-1)} - \frac{16}{27(z-2)} \right] dz \\
&= -\frac{17}{108} \ln|z+1| - \frac{5}{18(z+1)} - \frac{1}{6(z+1)^2} \\
&\quad + \frac{3}{4} \ln|z-1| - \frac{16}{27} \ln|z-2| + C,
\end{aligned}$$

其中 $z = \frac{\sqrt{x^2+3x+2}}{x+1}$

1970. $\int \frac{dx}{(1+\sqrt{x(1+x)})^2}$

解 设 $\sqrt{x(1+x)} = z+x$, 则

$$x = \frac{z^2}{1-2z}, \quad dx = \frac{2z(1-z)}{(1-2z)^2} dz,$$

$$1 + \sqrt{x(1+x)} = \frac{1-z-z^2}{1-2z}.$$

代入得

$$\begin{aligned}
\int \frac{dx}{(1+\sqrt{x(1+x)})^2} &= 2 \int \frac{z(1-z)}{(1-z-z^2)^2} dz \\
&= 2 \int \frac{(1-z-z^2) + (2z+1) - 2}{(1-z-z^2)^2} dz
\end{aligned}$$

$$\begin{aligned}
&= 2 \int \frac{dz}{1-z-z^2} - 2 \int \frac{d(1-z-z^2)}{(1-z-z^2)^2} \\
&\quad - 4 \int \frac{dz}{(1-z-z^2)^2} = 2 \int \frac{d\left(z+\frac{1}{2}\right)}{\frac{5}{4}-\left(z+\frac{1}{2}\right)^2} \\
&\quad + \frac{2}{1-z-z^2} - 4 \left\{ \frac{2z+1}{5(1-z-z^2)} + \frac{2}{5} \int \frac{d\left(z+\frac{1}{2}\right)}{\frac{5}{4}-\left(z+\frac{1}{2}\right)^2} \right\}^{*}) \\
&= \frac{2}{5\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}}{2} + z + \frac{1}{2}}{\frac{\sqrt{5}}{2} - z - \frac{1}{2}} \right| \\
&\quad + \frac{2}{1-z-z^2} - \frac{4(2z+1)}{5(1-z-z^2)} + C \\
&= \frac{2}{5\sqrt{5}} \ln \left| \frac{\sqrt{5} + 2z + 1}{\sqrt{5} - 2z - 1} \right| + \frac{2(3-4z)}{5(1-z-z^2)} + C,
\end{aligned}$$

其中 $z = \sqrt{x(1+x)} - x$.

*) 利用1921题的递推公式.

利用各种方法, 计算下列积分:

1971. $\int \frac{dx}{\sqrt{x^2+1} - \sqrt{x^2-1}}.$

$$\begin{aligned}
\text{解} \quad &\int \frac{dx}{\sqrt{x^2+1} - \sqrt{x^2-1}} = \int \frac{\sqrt{x^2+1} + \sqrt{x^2-1}}{(x^2+1) - (x^2-1)} dx \\
&= \frac{1}{2} \int \sqrt{x^2+1} dx + \frac{1}{2} \int \sqrt{x^2-1} dx
\end{aligned}$$

$$= \frac{x}{4} (\sqrt{x^2+1} + \sqrt{x^2-1}) + \frac{1}{4} \ln \left| \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2-1}} \right| + C.$$

1972. $\int \frac{x dx}{(1-x^3)\sqrt{1-x^2}}.$

解 设 $\frac{1+x}{1-x} = z$, 则

$$x = \frac{z-1}{z+1}, \quad dx = \frac{2}{(z+1)^2} dz,$$

代入得

$$\begin{aligned} \int \frac{x dx}{(1-x^3)\sqrt{1-x^2}} &= \frac{1}{2} \int \frac{(z^2-1) dz}{\sqrt{z}(3z^2+1)} \\ &= \int \frac{(z^2-1) d(\sqrt{z})}{3z^2+1} = \int \left[\frac{1}{3} - \frac{4}{3(3z^2+1)} \right] d(\sqrt{z}) \\ &= \frac{\sqrt{z}}{3} - \frac{4}{3} \cdot \frac{1}{\sqrt{3}} \int \frac{d(\sqrt{3z^2})}{(\sqrt{3z^2})^4+1} \\ &= \frac{\sqrt{z}}{3} - \frac{4}{3\sqrt{3}} \left[\frac{1}{4\sqrt{2}} \ln \frac{z\sqrt{3} + \sqrt{12z^2+1}}{z\sqrt{3} - \sqrt{12z^2+1}} \right. \\ &\quad \left. + \frac{1}{2\sqrt{2}} \operatorname{arctg} \left(\frac{\sqrt{12z^2}}{1-z\sqrt{3}} \right) \right]^* + C \\ &= \frac{\sqrt{z}}{3} - \frac{1}{3\sqrt{12}} \left[\ln \frac{z\sqrt{3} + \sqrt{12z^2+1}}{z\sqrt{3} - \sqrt{12z^2+1}} \right. \\ &\quad \left. - 2 \operatorname{arctg} \left(\frac{\sqrt{12z^2}}{z\sqrt{3}-1} \right) \right] + C, \end{aligned}$$

其中 $z = \frac{1+x}{1-x}$.

*) 利用1884题的结果。

$$1973. \int \frac{dx}{\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}$$

$$\text{解} \int \frac{dx}{\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}$$

$$\begin{aligned} &= \int \frac{-\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}{(\sqrt{2} + \sqrt{1-x} + \sqrt{1+x})(-\sqrt{2} + \sqrt{1-x} + \sqrt{1+x})} dx \\ &= \int \frac{-\sqrt{2} + \sqrt{1-x} + \sqrt{1+x}}{2\sqrt{1-x^2}} dx \\ &= -\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \int \frac{dx}{\sqrt{1+x}} + \frac{1}{2} \int \frac{dx}{\sqrt{1-x}} \\ &= -\frac{1}{\sqrt{2}} \arcsin x + \sqrt{1+x} - \sqrt{1-x} + C. \end{aligned}$$

$$1974. \int \frac{x + \sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}} dx.$$

$$\text{解} \int \frac{x + \sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}} dx$$

$$= \int \frac{(x + \sqrt{1+x+x^2})(1+x - \sqrt{1+x+x^2})}{(1+x)^2 - (1+x+x^2)} dx$$

$$= \int \frac{\sqrt{1+x+x^2} - 1}{x} dx$$

$$= \int \frac{\sqrt{1+x+x^2}}{x} dx - \ln|x|.$$

对于积分 $\int \frac{\sqrt{1+x+x^2}}{x} dx$, 设 $x = \frac{1}{t}$, 则

$$dx = -\frac{1}{t^2} dt, \quad \sqrt{1+x+x^2} = \frac{\sqrt{t^2+t+1}}{|t|}.$$

不妨设 $t > 0$, 代入得

$$\begin{aligned} \int \frac{\sqrt{1+x+x^2}}{x} dx &= -\int \frac{\sqrt{t^2+t+1}}{t^2} dt \\ &= \int \sqrt{t^2+t+1} d\left(\frac{1}{t}\right) \\ &= \frac{\sqrt{t^2+t+1}}{t} - \frac{1}{2} \int \frac{2t+1}{t\sqrt{1+t+t^2}} dt \\ &= \sqrt{x^2+x+1} - \int \frac{dt}{\sqrt{1+t+t^2}} - \frac{1}{2} \int \frac{dt}{t\sqrt{1+t+t^2}} \\ &= \sqrt{x^2+x+1} - \ln\left(t + \frac{1}{2} + \sqrt{1+t+t^2}\right) \\ &\quad + \frac{1}{2} \int \frac{d\left(\frac{1}{t}\right)}{\sqrt{\left(\frac{1}{t}\right)^2 + \left(\frac{1}{t}\right) + 1}} \\ &= \sqrt{x^2+x+1} - \ln \frac{2+x+2\sqrt{1+x+x^2}}{2x} \\ &\quad + \frac{1}{2} \ln\left(\frac{1}{t} + \frac{1}{2} + \sqrt{\frac{1}{t^2} + \frac{1}{t} + 1}\right) + C_1 \\ &= \sqrt{x^2+x+1} - \ln \frac{2+x+2\sqrt{1+x+x^2}}{2x} \\ &\quad + \frac{1}{2} \ln \frac{2x+1+2\sqrt{1+x+x^2}}{2} + C_1 \end{aligned}$$

$$= \sqrt{x^2 + x + 1} + \frac{1}{2} \ln \frac{2x + 1 + 2\sqrt{1 + x + x^2}}{(2 + x + 2\sqrt{1 + x + x^2})^2} + \ln x + C.$$

于是, 当 $x > 0$ 时, 最后得到

$$\int \frac{x + \sqrt{1 + x + x^2}}{1 + x + \sqrt{1 + x + x^2}} dx = \sqrt{x^2 + x + 1} + \frac{1}{2} \ln \frac{2x + 1 + 2\sqrt{1 + x + x^2}}{(2 + x + 2\sqrt{1 + x + x^2})^2} + C,$$

当 $x < 0$ 时, 可获同样的结果.

1975. $\int \frac{\sqrt{x(x+1)}}{\sqrt{x} + \sqrt{x+1}} dx.$

解 $\int \frac{\sqrt{x(x+1)}}{\sqrt{x} + \sqrt{x+1}} dx$

$$= \int \frac{\sqrt{x(x+1)} \cdot (\sqrt{x+1} - \sqrt{x})}{(x+1) - x} dx$$

$$= \int [(x+1)\sqrt{x} - x\sqrt{x+1}] dx$$

$$= \int \left[x^{\frac{3}{2}} + x^{\frac{1}{2}} - (x+1)^{\frac{3}{2}} + (x+1)^{\frac{1}{2}} \right] dx$$

$$= \frac{2}{3} \left[(x+1)^{\frac{3}{2}} + x^{\frac{3}{2}} \right] - \frac{2}{5} \left[(x+1)^{\frac{5}{2}} - x^{\frac{5}{2}} \right] + C.$$

1976. $\int \frac{(x^2-1)dx}{(x^2+1)\sqrt{x^4+1}}.$

解 $\int \frac{(x^2-1)dx}{(x^2+1)\sqrt{x^4+1}} = \int \frac{x^2-1}{\sqrt{\frac{x^4+1}{(x^2+1)^2}}}$

$$= \int \frac{x^2 - 1}{(x^2 + 1)^2} dx = \int \frac{dx}{\sqrt{1 - \left(\frac{x\sqrt{2}}{x^2 + 1}\right)^2}}.$$

下面我们先考虑积分 $\int \frac{x^2 - 1}{(x^2 + 1)^2} dx$. 设 $x = \operatorname{tg} t$,

$-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则有 $dx = \sec^2 t dt$.

代入得

$$\begin{aligned} \int \frac{x^2 - 1}{(x^2 + 1)^2} dx &= \int \frac{\operatorname{tg}^2 t - 1}{\sec^4 t} \cdot \sec^2 t dt \\ &= \int (\sin^2 t - \cos^2 t) dt = - \int \cos 2t dt \\ &= -\frac{1}{2} \sin 2t + C_1 = -\frac{x}{1 + x^2} + C_1, \end{aligned}$$

从而, 可得 $\frac{x^2 - 1}{(x^2 + 1)^2} dx = -\frac{1}{\sqrt{2}} d\left(\frac{x\sqrt{2}}{1 + x^2}\right)$.

于是,

$$\begin{aligned} \int \frac{(x^2 - 1) dx}{(x^2 + 1)\sqrt{x^4 + 1}} &= -\frac{1}{\sqrt{2}} \int \frac{d\left(\frac{x\sqrt{2}}{1 + x^2}\right)}{\sqrt{1 - \left(\frac{x\sqrt{2}}{1 + x^2}\right)^2}} \\ &= -\frac{1}{\sqrt{2}} \operatorname{arc} \sin\left(\frac{x\sqrt{2}}{1 + x^2}\right) + C. \end{aligned}$$

1977. $\int \frac{x^2 + 1}{(x^2 - 1)\sqrt{x^4 + 1}} dx.$

解 仿照1976题, 可得

$$\begin{aligned}
\int \frac{x^2+1}{(x^2-1)\sqrt{x^4+1}} dx &= \int \frac{\frac{x^2+1}{(x^2-1)^2}}{\sqrt{\frac{x^4+1}{(x^2-1)^2}}} dx \\
&= -\frac{1}{\sqrt{2}} \int \frac{d\left(\frac{x\sqrt{2}}{x^2-1}\right)}{\sqrt{1+\left(\frac{x\sqrt{2}}{x^2-1}\right)^2}} \\
&= -\frac{1}{\sqrt{2}} \ln \left| \frac{x\sqrt{2}}{x^2-1} + \sqrt{1+\left(\frac{x\sqrt{2}}{x^2-1}\right)^2} \right| + C \\
&= -\frac{1}{\sqrt{2}} \ln \left| \frac{x\sqrt{2} + \sqrt{x^4+1}}{x^2-1} \right| + C.
\end{aligned}$$

1978. $\int \frac{dx}{x\sqrt{x^4+2x^2-1}}$

解 作变换 $\frac{1}{x} = \sqrt{t}$ (这里设 $x > 0$, 若 $x < 0$, 则作变换 $\frac{1}{x} = -\sqrt{t}$, 最后结果相同), 则

$$dx = -\frac{1}{2t\sqrt{t}} dt, \sqrt{x^4+2x^2-1} = \frac{\sqrt{1+2t-t^2}}{t}$$

代入得

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^4+2x^2-1}} &= -\frac{1}{2} \int \frac{dt}{\sqrt{1+2t-t^2}} \\
&= \frac{1}{2} \int \frac{d(1-t)}{\sqrt{2-(1-t)^2}} \\
&= \frac{1}{2} \arcsin\left(\frac{1-t}{\sqrt{2}}\right) + C
\end{aligned}$$

$$= \frac{1}{2} \arcsin \left(\frac{x^2 - 1}{x^2 \sqrt{2}} \right) + C \quad (|x| > \sqrt{\sqrt{2} - 1}).$$

1979. $\int \frac{(x^2 + 1) dx}{x \sqrt{x^4 + x^2 + 1}}.$

解 $\int \frac{(x^2 + 1) dx}{x \sqrt{x^4 + x^2 + 1}} = \int \frac{x dx}{\sqrt{x^4 + x^2 + 1}} + \int \frac{dx}{x \sqrt{x^4 + x^2 + 1}}$

$$= \frac{1}{2} \int \frac{d\left(x^2 + \frac{1}{2}\right)}{\sqrt{\left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}}} - \frac{1}{2} \int \frac{d\left(\frac{1}{x^2}\right)}{\sqrt{\left(\frac{1}{x^2} + \frac{1}{2}\right)^2 + \frac{3}{4}}}$$

$$= \frac{1}{2} \ln \frac{x^2 + \frac{1}{2} + \sqrt{x^4 + x^2 + 1}}{\frac{1}{x^2} + \frac{1}{2} + \sqrt{\frac{x^4 + x^2 + 1}{x^4}}} + C$$

$$= \frac{1}{2} \ln \frac{x^2(1 + 2x^2 + 2\sqrt{x^4 + x^2 + 1})}{2 + x^2 + 2\sqrt{x^4 + x^2 + 1}} + C.$$

1980. 证明积分

$$\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$$

(式中 R 为有理函数) 的求法, 归结为有理函数的积分法.

证 当 a, c 中至少有一个为零时, 则积分

$$\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$$

的求法显然可归结为有理函数的积分法.

当 $a \neq 0, c \neq 0$ 时, 设 $\sqrt{ax+b} = z$, 则

$$x = \frac{z^2 - b}{a}, \quad dx = \frac{2}{a} z dz,$$

$$\sqrt{cx + d} = \sqrt{\frac{c}{a} z^2 + d - \frac{bc}{a}} = \sqrt{c_1 z^2 + d_1},$$

式中 $c_1 = \frac{c}{a}$, $d_1 = d - \frac{bc}{a}$.

代入得

$$\begin{aligned} & \int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx \\ &= \int R\left(\frac{z^2 - b}{a}, z, \sqrt{c_1 z^2 + d_1}\right) \frac{2}{a} z dz, \\ &= \int R_1(z, \sqrt{c_1 z^2 + d_1}) dz, \end{aligned}$$

其中 R_1 为有理函数.

再设 $\sqrt{c_1 z^2 + d_1} = \pm \sqrt{c_1} z + u$ ($c_1 > 0$) 或 $\sqrt{c_1 z^2 + d_1} = zu \pm \sqrt{d_1}$ ($d_1 > 0$) —— 尤拉代换, 就可将被积函数有理化. 于是, 积分

$$\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx$$

的求法可归结为有理函数的积分法.

二项微分式

$$\int x^m (a + bx^n)^p dx,$$

(式中 m , n 和 p 为有理数) 仅在下列三种情形可化为有理函数的积分 (契比协夫定理):

第一种情形, p 为整数. 假定 $x = z^N$, 其中 N 为分数 m 和 n 的公分母.

第二种情形, $\frac{m+1}{n}$ 为整数. 假定 $a + bx^n = z^N$, 其中 N 为分数 p 的分母.

第三种情形, $\frac{m+1}{n} + p$ 为整数. 利用代换: $ax^{-n} + b = z^N$, 其中 N 为分数 p 的分母. 计算下列积分:

$$1981. \int \sqrt{x^3 + x^4} dx.$$

解 $\sqrt{x^3 + x^4} = x^{\frac{3}{2}}(1+x)^{\frac{1}{2}}$. $m = \frac{3}{2}$, $n = 1$, $p = \frac{1}{2}$;

$\frac{m+1}{n} + p = 3$, 这是二项微分式的第三种情形.

设 $x^{-1} + 1 = z^2$, 则

$$x = \frac{1}{z^2 - 1}, \quad dx = -\frac{2z}{(z^2 - 1)^2} dz,$$

$$\sqrt{x^3 + x^4} = \frac{z}{(z^2 - 1)^2} \quad (\text{不妨设 } z > 0, \text{ 以下各题}$$

不再说明).

代入得

$$\begin{aligned} \int \sqrt{x^3 + x^4} dx &= -2 \int \frac{z^2}{(z^2 - 1)^4} dz \\ &= -2 \int \frac{dz}{(z^2 - 1)^4} = -2 \int \frac{dz}{(z^2 - 1)^3} \end{aligned}$$

$$\begin{aligned}
&= -2 \left[-\frac{z}{6(z^2-1)^3} - \frac{5}{6} \int \frac{dz}{(z^2-1)^3} \right]^{*}) - 2 \int \frac{dz}{(z^2-1)^3} \\
&= \frac{z}{3(z^2-1)^3} - \frac{1}{3} \int \frac{dz}{(z^2-1)^3} \\
&= \frac{z}{3(z^2-1)^3} + \frac{z}{12(z^2-1)^2} - \frac{z}{8(z^2-1)} \\
&\quad + \frac{1}{16} \ln \frac{z+1}{z-1} + C \\
&= \frac{1}{3} \sqrt{(x+x^2)^3} - \frac{1+2x}{8} \sqrt{x+x^2} \\
&\quad + \frac{1}{8} \ln(\sqrt{x} + \sqrt{1+x}) + C \quad (x > 0).
\end{aligned}$$

*) 利用1921题的结果。

1982. $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2} dx.$

解 $\frac{\sqrt{x}}{(1+\sqrt[3]{x})^2} = x^{\frac{1}{2}}(1+x^{\frac{1}{3}})^{-2}$. $m = \frac{1}{2}$, $n = \frac{1}{3}$, $p =$

-2 ; p 为整数, 这是二项微分式的第一种情形.

设 $x = z^6$, 则

$$dx = 6z^5 dz, \quad \sqrt{x} = z^3, \quad \sqrt[3]{x} = z^2.$$

代入得

$$\begin{aligned}
\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2} dx &= 6 \int \frac{z^3}{(z^2+1)^2} dz \\
&= 6 \int \left[z^4 - 2z^2 + 3 - \frac{4}{z^2+1} + \frac{1}{(z^2+1)^2} \right] dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{6}{5}z^5 - 4z^3 + 18z - 24 \operatorname{arc} \operatorname{tg} z \\
&+ 6 \left[\frac{z}{2(z^2+1)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} z \right]^{*}) + C \\
&= \frac{6}{5}x^{\frac{5}{3}} - 4x^{\frac{1}{2}} + 18x^{\frac{1}{6}} + \frac{3x^{\frac{1}{6}}}{1+x^{\frac{1}{3}}} - 21 \operatorname{arc} \operatorname{tg}(x^{\frac{1}{6}}) + C.
\end{aligned}$$

*) 利用 1921 题的结果.

1983. $\int \frac{x dx}{\sqrt{1+\sqrt[3]{x^2}}}.$

解 $\frac{x}{\sqrt{1+\sqrt[3]{x^2}}} = x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}, m=1, n=\frac{2}{3}, p=-\frac{1}{2};$

$\frac{m+1}{n}=3$, 这是二项微分式的第二种情形.

设 $1+x^{\frac{2}{3}}=z^2$, 则

$$x=(z^2-1)^{\frac{3}{2}}, \quad dx=3z(z^2-1)^{\frac{1}{2}}dz.$$

代入得

$$\int \frac{x dx}{\sqrt{1+\sqrt[3]{x^2}}} = 3 \int (z^2-1)^2 dz$$

$$= \frac{3}{5}z^5 - 2z^3 + 3z + C,$$

其中 $z = \sqrt{1+\sqrt[3]{x^2}}$.

1984. $\int \frac{x^5 dx}{\sqrt{1-x^2}}$

解 $\frac{x^5}{\sqrt{1-x^2}} = x^5(1-x^2)^{-\frac{1}{2}}, m=5, n=2, p=-\frac{1}{2};$

$\frac{m+1}{n} = 3$, 这是二项微分式的第二种情形.

设 $\sqrt{1-x^2} = z$ (不妨设 $x > 0$), 则

$$x = \sqrt{1-z^2}, \quad dx = -\frac{z}{\sqrt{1-z^2}} dz.$$

代入得

$$\int \frac{x^5 dx}{\sqrt{1-x^2}} = -\int (1-z^2)^2 dz$$

$$= -z + \frac{2}{3}z^3 - \frac{1}{5}z^5 + C,$$

其中 $z = \sqrt{1-x^2}$.

1985. $\int \frac{dx}{\sqrt[3]{1+x^3}}$.

解 $\frac{1}{\sqrt[3]{1+x^3}} = x^0(1+x^3)^{-\frac{1}{3}}$, $m=0, n=3, p=-\frac{1}{3}$;

$\frac{m+1}{n} + p = 0$, 这是二项微分式的第三种情形.

设 $x^{-3} + 1 = z^3$, 则

$$x = (z^3 - 1)^{-\frac{1}{3}}, \quad dx = -z^2(z^3 - 1)^{-\frac{4}{3}} dz.$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[3]{1+x^3}} &= -\int \frac{z}{z^3-1} dz \\ &= -\frac{1}{3} \int \frac{dz}{z-1} + \frac{1}{3} \int \frac{z-1}{z^2+z+1} dz \\ &= -\frac{1}{3} \ln|z-1| + \frac{1}{6} \ln(z^2+z+1) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\sqrt{3}} \operatorname{arctg} \left(\frac{2z+1}{\sqrt{3}} \right) + C \\
& = \frac{1}{6} \ln \frac{z^2+z+1}{(z-1)^2} - \frac{1}{\sqrt{3}} \operatorname{arctg} \left(\frac{2z+1}{\sqrt{3}} \right) + C,
\end{aligned}$$

其中 $z = \frac{\sqrt[3]{1+x^3}}{x}$.

1986. $\int \frac{dx}{\sqrt[4]{1+x^4}}$.

解 $\frac{1}{\sqrt[4]{1+x^4}} = x^0(1+x^4)^{-\frac{1}{4}}$, $m=0, n=4, p=-\frac{1}{4}$;

$\frac{m+1}{n} + p = 0$, 这是二项微分式的第三种情形.

设 $x^{-4} + 1 = z^4$, 则

$$z = \frac{\sqrt[4]{1+x^4}}{x} \quad (z > 0, x > 0),$$

$$x = (z^4 - 1)^{-\frac{1}{4}}, \quad dx = -z^3(z^4 - 1)^{-\frac{5}{4}} dz.$$

代入得

$$\begin{aligned}
\int \frac{dx}{\sqrt[4]{1+x^4}} &= -\int \frac{z^2}{z^4-1} dz \\
&= \int \left[\frac{1}{4(z+1)} - \frac{1}{4(z-1)} - \frac{1}{2(z^2+1)} \right] dz \\
&= \frac{1}{4} \ln \left| \frac{z+1}{z-1} \right| - \frac{1}{2} \operatorname{arc.} \operatorname{tg} z + C,
\end{aligned}$$

其中 $z = \frac{\sqrt[6]{1+x^6}}{x}$.

1987⁺. $\int \frac{dx}{x\sqrt[6]{1+x^6}}$.

解 $\frac{1}{x\sqrt[6]{1+x^6}} = x^{-1}(1+x^6)^{-\frac{1}{6}}$, $m = -1$, $n = 6$, p

$= -\frac{1}{6}$; $\frac{m+1}{n} = 0$, 这是二项微分式的第二种情形.

设 $1+x^6 = z^6$, 则

$z = \sqrt[6]{1+x^6}$ ($z > 0, x > 0$),

$x = \sqrt[6]{z^6-1}$, $dx = z^5(z^6-1)^{-\frac{5}{6}}dz$.

代入得

$$\int \frac{dx}{x\sqrt[6]{1+x^6}} = \int \frac{z^4 dz}{z^6-1}$$

$$= \int \left[-\frac{1}{6(z+1)} + \frac{z+1}{6(z^2-z+1)} \right.$$

$$\left. + \frac{1}{6(z-1)} + \frac{-z+1}{6(z^2+z+1)} \right] dz$$

$$= \frac{1}{6} \ln \frac{z-1}{z+1} + \frac{1}{12} \ln \frac{z^2-z+1}{z^2+z+1}$$

$$+ \frac{1}{2\sqrt{3}} \left[\operatorname{arctg} \left(\frac{2z-1}{\sqrt{3}} \right) + \operatorname{arctg} \left(\frac{2z+1}{\sqrt{3}} \right) \right] + C_1$$

$$= \frac{1}{6} \ln \frac{z-1}{z+1} + \frac{1}{12} \ln \frac{z^2-z+1}{z^2+z+1}$$

$$+\frac{1}{2\sqrt{\frac{2}{3}}}\arctg\left(\frac{z^2-1}{z\sqrt{\frac{2}{3}}}\right)+C,$$

其中 $z = \sqrt[6]{1+x^6}$.

1988.
$$\int \frac{dx}{x^3 \sqrt[5]{1+\frac{1}{x}}}.$$

解
$$\frac{1}{x^3 \sqrt[5]{1+\frac{1}{x}}} = x^{-3}(1+x^{-1})^{-\frac{1}{5}}, m = -3, n = -1,$$

$p = -\frac{1}{5}; \frac{m+1}{n} = 2$, 这是二项微分式的第二种情形.

设 $1+x^{-1} = z^5$, 则

$$x = (z^5 - 1)^{-1}, dx = -5z^4(z^5 - 1)^{-2} dz.$$

代入得

$$\begin{aligned} \int \frac{dx}{x^3 \sqrt[5]{1+\frac{1}{x}}} &= -5 \int z^3(z^5 - 1) dz \\ &= -\frac{5}{9}z^9 + \frac{5}{4}z^4 + C, \end{aligned}$$

其中 $z = \sqrt[5]{1+\frac{1}{x}}$.

1989.
$$\int \sqrt[3]{3x-x^3} dx.$$

解
$$\sqrt[3]{3x-x^3} = x^{\frac{1}{3}}(3-x^2)^{\frac{1}{3}}, m = \frac{1}{3}, n = 2, p = \frac{1}{3},$$

$\frac{m+1}{n} + p = 1$, 这是二项微分式的第三种情形.

设 $3x^2 - 1 = z^3$ (不妨设 $x > 0$) , 则

$$z = \frac{\sqrt[3]{3x - x^3}}{x}, \quad x = \frac{\sqrt[3]{3}}{\sqrt{z^3 + 1}}$$

$$dx = -\frac{3\sqrt[3]{3}}{2} \cdot \frac{z^2}{(z^3 + 1)^2} dz.$$

代入得

$$\begin{aligned} \int \sqrt[3]{3x - x^3} dx &= -\frac{9}{2} \int \frac{z^2}{(z^3 + 1)^2} dz \\ &= -\frac{9}{2} \int \frac{dz}{z^3 + 1} + \frac{9}{2} \int \frac{dz}{(z^3 + 1)^2} \\ &= -\frac{9}{2} \left[\frac{1}{6} \ln \frac{(z+1)^2}{z^2 - z + 1} + \frac{1}{\sqrt[3]{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2z-1}{\sqrt[3]{3}} \right) \right]^{*}) \\ &+ \frac{9}{2} \left[\frac{z}{3(z^3 + 1)} + \frac{1}{9} \ln \frac{(z+1)^2}{z^2 - z + 1} \right. \\ &\left. + \frac{2}{3\sqrt[3]{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2z-1}{\sqrt[3]{3}} \right) \right]^{**)} + C \\ &= \frac{3z}{2(z^3 + 1)} - \frac{1}{4} \ln \frac{(z+1)^2}{z^2 - z + 1} \\ &- \frac{\sqrt[3]{3}}{2} \operatorname{arc} \operatorname{tg} \left(\frac{2z-1}{\sqrt[3]{3}} \right) + C, \end{aligned}$$

其中 $z = \frac{\sqrt[3]{3x - x^3}}{x}$.

*) 利用1881题的结果.

**) 利用1892题的结果.

1990. 在甚么情形下, 积分

$$\int \sqrt{1+x^m} dx$$

(式中 m 为有理数) 为初等函数?

解 $\sqrt{1+x^m} = x^0(1+x^m)^{\frac{1}{2}}$. 由于 $p = \frac{1}{2}$, 故由契比协夫定理知, 仅在下述两种情形, 此函数的积分可化为有理函数的积分.

第一种情形, $\frac{1}{m}$ 为整数, 即 $m = \frac{1}{k_1} = \frac{2}{2k_1}$, 其中 $k_1 = \pm 1, \pm 2, \dots$;

第二种情形, $\frac{1}{m} + \frac{1}{2}$ 为整数, 即 $m = \frac{2}{2k_2 - 1}$, 其中 $k_2 = 0, \pm 1, \pm 2, \dots$.

综上所述, 即得: 当

$$m = \frac{2}{k}$$

(式中 $k = \pm 1, \pm 2, \dots$) 时, 积分

$$\int \sqrt{1+x^m} dx$$

为初等函数.

§4. 三角函数的积分法

形如

$$\int \sin^m x \cos^n x dx$$

的积分 (式中 m 及 n 为整数), 可利用巧妙的变换或

运用递推公式计算。

求下列积分：

$$1991. \int \cos^5 x dx.$$

$$\begin{aligned} \text{解} \quad \int \cos^5 x dx &= \int \cos^4 x \cos x dx \\ &= \int (1 - \sin^2 x)^2 d(\sin x) \\ &= \int (1 - 2\sin^2 x + \sin^4 x) d(\sin x) \\ &= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C. \end{aligned}$$

$$1992. \int \sin^6 x dx.$$

$$\begin{aligned} \text{解} \quad \int \sin^6 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^3 dx \\ &= \frac{1}{8} \int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x) dx \\ &= \frac{x}{8} - \frac{3}{16} \sin 2x + \frac{3}{8} \int \frac{1 + \cos 4x}{2} dx \\ &\quad - \frac{1}{8} \int (1 - \sin^2 2x) \cos 2x dx \\ &= \frac{x}{8} - \frac{3}{16} \sin 2x + \frac{3x}{16} + \frac{3}{64} \sin 4x \\ &\quad - \frac{1}{16} \int (1 - \sin^2 2x) d(\sin 2x) \\ &= \frac{5x}{16} - \frac{3}{16} \sin 2x + \frac{3}{64} \sin 4x \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{16}\sin 2x + \frac{1}{48}\sin^3 2x + C \\
& = \frac{5x}{16} - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

1993. $\int \cos^6 x dx.$

$$\begin{aligned}
\text{解} \quad \int \cos^6 x dx &= \int \sin^6\left(x - \frac{\pi}{2}\right) d\left(x - \frac{\pi}{2}\right) \\
&= \frac{5}{16}\left(x - \frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(x - \frac{\pi}{2}\right) + \frac{3}{64}\sin 4\left(x - \frac{\pi}{2}\right) \\
&\quad + \frac{1}{48}\sin^3 2\left(x - \frac{\pi}{2}\right)^* + C_1 \\
&= \frac{5x}{16} + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

*) 利用1992题的结果。

1994. $\int \sin^2 x \cos^4 x dx.$

$$\begin{aligned}
\text{解} \quad \int \sin^2 x \cos^4 x dx &= \frac{1}{4} \int \sin^2 2x \cos^2 x dx \\
&= \frac{1}{8} \int \sin^2 2x (1 + \cos 2x) dx \\
&= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx + \frac{1}{16} \int \sin^2 2x d(\sin 2x) \\
&= \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.
\end{aligned}$$

1995. $\int \sin^4 x \cos^5 x dx.$

$$\begin{aligned} \text{解} \quad \int \sin^4 x \cos^5 x dx &= \int \sin^4 x (1 - \sin^2 x)^2 d(\sin x) \\ &= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C. \end{aligned}$$

$$1996. \quad \int \sin^5 x \cos^5 x dx.$$

$$\begin{aligned} \text{解} \quad \int \sin^5 x \cos^5 x dx &= \frac{1}{32} \int \sin^5 2x dx \\ &= -\frac{1}{64} \int (1 - \cos^2 2x)^2 d(\cos 2x) \\ &= -\frac{1}{64} \cos 2x + \frac{1}{96} \cos^3 2x - \frac{1}{320} \cos^5 2x + C. \end{aligned}$$

$$1997. \quad \int \frac{\sin^3 x}{\cos^4 x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin^3 x}{\cos^4 x} dx &= - \int \frac{1 - \cos^2 x}{\cos^4 x} d(\cos x) \\ &= -\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C \end{aligned}$$

$$1998. \quad \int \frac{\cos^4 x}{\sin^3 x} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\cos^4 x}{\sin^3 x} dx &= \int \frac{\cos^3 x}{\sin^3 x} d(\sin x) \\ &= -\frac{1}{2} \int \cos^3 x d\left(-\frac{1}{\sin^2 x}\right) \\ &= -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2} \int \frac{\cos^2 x \sin x}{\sin^2 x} dx \end{aligned}$$

$$\begin{aligned}
 &= -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2} \int \frac{1 - \sin^2 x}{\sin x} dx \\
 &= -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right| - \frac{3}{2} \cos x + C.
 \end{aligned}$$

1999. $\int \frac{dx}{\sin^3 x}$

解 $\int \frac{dx}{\sin^3 x} = -\int \frac{1}{\sin x} d(\operatorname{ctg} x) = -\frac{\operatorname{ctg} x}{\sin x}$
 $= -\int \operatorname{ctg} x \frac{\cos x}{\sin^2 x} dx = -\frac{\cos x}{\sin^2 x} - \int \frac{1 - \sin^2 x}{\sin^3 x} dx$
 $= -\frac{\cos x}{\sin^2 x} - \int \frac{dx}{\sin^3 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|,$

于是,

$$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.$$

2000. $\int \frac{dx}{\cos^3 x}$

解 $\int \frac{dx}{\cos^3 x} = \int \frac{d\left(x + \frac{\pi}{2}\right)}{\sin^3\left(x + \frac{\pi}{2}\right)}$
 $= -\frac{\cos\left(x + \frac{\pi}{2}\right)}{2\sin^2\left(x + \frac{\pi}{2}\right)} + \frac{1}{2} \ln \left| \operatorname{tg}\left(\frac{x + \frac{\pi}{2}}{2}\right) \right| + C$
 $= \frac{\sin x}{2\cos^2 x} + \frac{1}{2} \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C.$

*) 利用1999题的结果。

$$2001. \int \frac{dx}{\sin^4 x \cos^4 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin^4 x \cos^4 x} &= 16 \int \frac{dx}{\sin^4 2x} \\ &= -8 \int \csc^2 2x d(\operatorname{ctg} 2x) \\ &= -8 \int (1 + \operatorname{ctg}^2 2x) d(\operatorname{ctg} 2x) \\ &= -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x + C. \end{aligned}$$

$$2002. \int \frac{dx}{\sin^3 x \cos^5 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{\sin^3 x \cos^5 x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos^5 x} dx \\ &= \int \frac{dx}{\sin x \cos^5 x} + \int \frac{dx}{\sin^3 x \cos^3 x} \\ &= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^5 x} dx + \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos^3 x} dx \\ &= \int \frac{\sin x}{\cos^5 x} dx + 2 \int \frac{dx}{\sin x \cos^3 x} + \int \frac{dx}{\sin^3 x \cos x} \\ &= - \int \frac{d(\cos x)}{\cos^5 x} + 2 \int \frac{\sin x}{\cos^3 x} dx \\ &\quad + 3 \int \frac{dx}{\sin x \cos x} + \int \frac{\cos x}{\sin^3 x} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\cos^4 x} - 2 \int \frac{d(\cos x)}{\cos^3 x} + 3 \int \frac{d(\operatorname{tg} x)}{\operatorname{tg} x} + \int \frac{d(\sin x)}{\sin^3 x} \\
&= \frac{1}{4\cos^4 x} + \frac{1}{\cos^2 x} + 3\ln|\operatorname{tg} x| - \frac{1}{2\sin^2 x} + C_1 \\
&= \frac{1}{4}\operatorname{tg}^4 x + \frac{3}{2}\operatorname{tg}^2 x - \frac{1}{2}\operatorname{ctg}^2 x + 3\ln|\operatorname{tg} x| + C.
\end{aligned}$$

2003. $\int \frac{dx}{\sin x \cos^4 x}$.

$$\begin{aligned}
\text{解} \quad &\int \frac{dx}{\sin x \cos^4 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^4 x} dx \\
&= \int \frac{\sin x}{\cos^4 x} dx + \int \frac{dx}{\sin x \cos^2 x} \\
&= - \int \frac{d(\cos x)}{\cos^4 x} + \int \frac{\sin x}{\cos^2 x} dx + \int \frac{dx}{\sin x} \\
&= \frac{1}{3\cos^3 x} - \int \frac{d(\cos x)}{\cos^2 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right| \\
&= \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.
\end{aligned}$$

2004. $\int \operatorname{tg}^5 x dx$.

$$\begin{aligned}
\text{解} \quad &\int \operatorname{tg}^5 x dx = \int \operatorname{tg} x (\sec^2 x - 1)^2 dx \\
&= \int \sec^4 x \operatorname{tg} x dx - 2 \int \sec^2 x \operatorname{tg} x dx + \int \operatorname{tg} x dx \\
&= \int \sec^3 x d(\sec x) - 2 \int \sec x d(\sec x) - \int \frac{d(\cos x)}{\cos x}
\end{aligned}$$

$$= \frac{1}{4} \sec^4 x - \sec^2 x - \ln |\cos x| + C_1$$

$$= \frac{1}{4} \operatorname{tg}^4 x - \frac{1}{2} \operatorname{tg}^2 x - \ln |\cos x| + C.$$

2005. $\int \operatorname{ctg}^6 x dx.$

$$\begin{aligned} \text{解} \quad \int \operatorname{ctg}^6 x dx &= \int \operatorname{ctg}^2 x (\operatorname{csc}^2 x - 1)^2 dx \\ &= \int \operatorname{ctg}^2 x \operatorname{csc}^4 x dx - 2 \int \operatorname{ctg}^2 x \operatorname{csc}^2 x dx + \int \operatorname{ctg}^2 x dx \\ &= - \int \operatorname{ctg}^2 x (1 + \operatorname{ctg}^2 x) d(\operatorname{ctg} x) \\ &\quad + 2 \int \operatorname{ctg}^2 x d(\operatorname{ctg} x) + \int (\operatorname{csc}^2 x - 1) dx \\ &= -\frac{1}{3} \operatorname{ctg}^3 x - \frac{1}{5} \operatorname{ctg}^5 x + \frac{2}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x - x + C \\ &= -\frac{1}{5} \operatorname{ctg}^5 x + \frac{1}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x - x + C. \end{aligned}$$

2006. $\int \frac{\sin^4 x}{\cos^8 x} dx.$

$$\text{解} \quad \int \frac{\sin^4 x}{\cos^8 x} dx = \int \operatorname{tg}^4 x d(\operatorname{tg} x) = \frac{1}{5} \operatorname{tg}^5 x + C.$$

2007. $\int \frac{dx}{\sqrt{\sin^3 x \cos^6 x}}.$

$$\text{解} \quad \int \frac{dx}{\sqrt{\sin^3 x \cos^6 x}} = \int \frac{\sin^2 x dx}{\sqrt{\sin^3 x \cos^5 x}}$$

$$\begin{aligned}
& + \int \frac{\cos^2 x dx}{\sqrt{\sin^3 x \cos^5 x}} \\
& = \int \sqrt{\operatorname{tg} x} d(\operatorname{tg} x) - \int \frac{d(\operatorname{ctg} x)}{\sqrt{\operatorname{ctg} x}} \\
& = \frac{2}{3} \sqrt{\operatorname{tg}^3 x} - 2\sqrt{\operatorname{ctg} x} + C.
\end{aligned}$$

2008⁺ $\int \frac{dx}{\cos x \sqrt{\sin^2 x}}$

解 设 $t = \sqrt[3]{\sin x}$, 不妨只考虑 $\cos x$ 为正的情况,

即 $-\frac{\pi}{2} < x < \frac{\pi}{2}$ 且 $x \neq 0$, 则有

$$dx = \frac{3t^2}{\sqrt{1-t^6}} dt, \quad \cos x = \sqrt{1-t^6}.$$

代入得

$$\begin{aligned}
& \int \frac{dx}{\cos x \sqrt{\sin^2 x}} = 3 \int \frac{dt}{1-t^6} \\
& = \frac{3}{2} \int \left(\frac{1}{1-t^3} + \frac{1}{1+t^3} \right) dt \\
& = \frac{1}{2} \int \left(\frac{1}{1-t} + \frac{t+2}{1+t+t^2} \right) dt + \frac{3}{2} \int \frac{dt}{1-t^3} \\
& = \frac{-1}{2} \ln|1-t| + \frac{1}{4} \int \frac{2t+1}{t^2+t+1} + \frac{3}{4} \int \frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} \\
& + \frac{3}{2} \left[\frac{1}{6} \ln \frac{(t+1)^2}{t^2-t+1} + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2t-1}{\sqrt{3}} \right] + C
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \ln \frac{(t+1)^2(t^2+t+1)}{(1-t)^2(t^2-t+1)} \\
&+ \frac{\sqrt{3}}{2} \left[\operatorname{arc} \operatorname{tg} \left(\frac{2t+1}{\sqrt{3}} \right) + \operatorname{arc} \operatorname{tg} \left(\frac{2t-1}{\sqrt{3}} \right) \right] + C \\
&= \frac{1}{4} \ln \frac{(1+t)^3(1-t^3)}{(1-t)^3(1+t^3)} + \frac{\sqrt{3}}{2} \operatorname{arc} \operatorname{tg} \left(\frac{t\sqrt{3}}{1-t^2} \right) + C,
\end{aligned}$$

其中 $t = \sqrt[3]{\sin x}$.

*) 利用1881题的结果.

2009. $\int \frac{dx}{\sqrt{\operatorname{tg} x}}$.

解 设 $t = \sqrt{\operatorname{tg} x}$, 则

$$x = \operatorname{arc} \operatorname{tg} t^2, \quad dx = \frac{2t}{1+t^4} dt,$$

代入得

$$\begin{aligned}
\int \frac{dx}{\sqrt{\operatorname{tg} x}} &= 2 \int \frac{dt}{1+t^4} \\
&= 2 \left(\frac{1}{4\sqrt{2}} \ln \frac{t^2+t\sqrt{2}+1}{t^2-t\sqrt{2}+1} \right. \\
&\quad \left. + \frac{1}{2\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{t\sqrt{2}}{1-t^2} \right)^* + C \\
&= \frac{1}{2\sqrt{2}} \ln \frac{t^2+t\sqrt{2}+1}{t^2-t\sqrt{2}+1} + \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{t\sqrt{2}}{1-t^2} + C,
\end{aligned}$$

其中 $t = \sqrt{\operatorname{tg} x}$.

*) 利用1884题的结果.

2010. $\int \frac{dx}{\sqrt[3]{\operatorname{tg}x}}$.

解 设 $\sqrt[3]{\operatorname{tg}x}=t$, 则

$$x = \operatorname{arc} \operatorname{tg} t^3, \quad dx = \frac{3t^2}{1+t^6} dt,$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[3]{\operatorname{tg}x}} &= 3 \int \frac{tdt}{1+t^6} = \frac{3}{2} \int \frac{d(t^2)}{1+(t^2)^3} \\ &= \frac{3}{2} \left[\frac{1}{6} \ln \frac{(t^2+1)^2}{t^4-t^2+1} - \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2t^2-1}{\sqrt{3}} \right]^* + C \\ &= \frac{1}{4} \ln \frac{(t^2+1)^2}{t^4-t^2+1} + \frac{\sqrt{3}}{2} \operatorname{arc} \operatorname{tg} \frac{2t^2-1}{\sqrt{3}} + C, \end{aligned}$$

其中 $t = \sqrt[3]{\operatorname{tg}x}$.

*) 利用1881题的结果.

2011. 推出下列积分的递推公式

$$(a) I_n = \int \sin^n x dx; \quad (b) K_n = \int \cos^n x dx \quad (n > 2).$$

并利用推得的公式来计算

$$\int \sin^6 x dx \text{ 及 } \int \cos^8 x dx.$$

$$\begin{aligned} \text{解 (a)} \quad I_n &= \int \sin^n x dx = - \int \sin^{n-1} x d(\cos x) \\ &= -\cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x dx \\ &= -\cos x \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\ &= -\cos x \sin^{n-1} x + (n-1) I_{n-2} + (1-n) I_n, \end{aligned}$$

于是,

$$I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

利用此公式及

$$I_0 = \int dx = x + C,$$

即得

$$\begin{aligned} I_6 &= \int \sin^6 x dx = -\frac{\cos x \sin^5 x}{6} + \frac{5}{6} I_4 \\ &= -\frac{\cos x \sin^5 x}{6} - \frac{5 \cos x \sin^3 x}{24} + \frac{5}{8} I_2 \\ &= -\frac{\cos x \sin^5 x}{6} - \frac{5 \cos x \sin^3 x}{24} \\ &\quad - \frac{5 \cos x \sin x}{16} + \frac{5}{16} x + C. \end{aligned}$$

$$\begin{aligned} (6) \quad K_n &= \int \cos^n x dx = \int \cos^{n-1} x d(\sin x) \\ &= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x dx \\ &= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx \\ &= \sin x \cos^{n-1} x + (n-1) K_{n-2} - (n-1) K_n \end{aligned}$$

于是,

$$K_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} K_{n-2}$$

利用此公式及

$$K_0 = x + C$$

即得

$$\begin{aligned} K_8 &= \int \cos^8 x dx = \frac{1}{8} \sin x \cos^7 x + \frac{7}{8} K_6 = \dots \\ &= \frac{1}{8} \sin x \cos^7 x + \frac{7}{48} \sin x \cos^5 x + \frac{35}{192} \sin x \cos^3 x \\ &\quad + \frac{35}{128} \sin x \cos x + \frac{35}{128} x + C. \end{aligned}$$

2012. 推出下列积分的递推公式

$$(a) I_n = \int \frac{dx}{\sin^n x}; \quad (b) K_n = \int \frac{dx}{\cos^n x} \quad (n > 2).$$

并利用推得的公式计算

$$\int \frac{dx}{\sin^5 x} \quad \text{及} \quad \int \frac{dx}{\cos^7 x}.$$

$$\begin{aligned} \text{解} \quad (a) \quad I_n &= \int \frac{dx}{\sin^n x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^n x} dx \\ &= I_{n-2} - \frac{1}{n-1} \int \cos x d\left(\frac{1}{\sin^{n-1} x}\right) \\ &= I_{n-2} - \frac{\cos x}{(n-1)\sin^{n-1} x} - \frac{1}{n-1} I_{n-2} \\ &= -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}. \end{aligned}$$

利用此公式及

$$I_1 = \int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C,$$

即得

$$I_5 = \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} + \frac{3}{4} I_3 = \dots$$

$$= -\frac{\cos x}{4\sin^4 x} - \frac{3\cos x}{8\sin^2 x} + \frac{3}{8} \ln \left| \operatorname{tg} \frac{x}{2} \right| + C.$$

$$(6) K_n = \int \frac{dx}{\cos^n x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^n x} dx$$

$$= \frac{1}{n-1} \int \sin x d\left(\frac{1}{\cos^{n-1} x}\right) + K_{n-2}$$

$$= \frac{\sin x}{(n-1)\cos^{n-1} x} - \frac{1}{n-1} K_{n-2} + K_{n-2}$$

$$= \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} K_{n-2};$$

利用此公式及

$$K_1 = \int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C,$$

即得

$$K_7 = \int \frac{dx}{\cos^7 x} = \frac{\sin x}{6\cos^6 x} + \frac{5}{6} K_5 = \dots$$

$$= \frac{\sin x}{6\cos^6 x} + \frac{5\sin x}{24\cos^4 x} + \frac{5\sin x}{16\cos^2 x}$$

$$+ \frac{5}{16} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C.$$

运用公式

$$\text{I} \quad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)),$$

$$\text{II} \quad \cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta)),$$

$$\text{III} \quad \sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$$

来计算下列的积分。

求积分：

$$2013. \quad \int \sin 5x \cos x dx.$$

$$\begin{aligned} \text{解} \quad \int \sin 5x \cos x dx &= \frac{1}{2} \int (\sin 4x + \sin 6x) dx \\ &= -\frac{1}{8} \cos 4x - \frac{1}{12} \cos 6x + C. \end{aligned}$$

$$2014. \quad \int \cos x \cos 2x \cos 3x dx.$$

$$\begin{aligned} \text{解} \quad \int \cos x \cos 2x \cos 3x dx &= \frac{1}{2} \int \cos 2x (\cos 4x + \cos 2x) dx \\ &= \frac{1}{4} \int (\cos 6x + \cos 2x) dx + \frac{1}{4} \int (1 + \cos 4x) dx \\ &= \frac{1}{24} \sin 6x + \frac{1}{8} \sin 2x + \frac{1}{16} \sin 4x + \frac{x}{4} + C. \end{aligned}$$

$$2015. \quad \int \sin x \sin \frac{x}{2} \sin \frac{x}{3} dx.$$

$$\begin{aligned}
\text{解} \quad & \int \sin x \sin \frac{x}{2} \sin \frac{x}{3} dx \\
&= \frac{1}{2} \int \left(\cos \frac{2}{3} x - \cos \frac{4}{3} x \right) \sin \frac{x}{2} dx \\
&= \frac{1}{2} \int \cos \frac{2}{3} x \sin \frac{x}{2} dx - \frac{1}{2} \int \cos \frac{4}{3} x \sin \frac{x}{2} dx \\
&= \frac{1}{4} \int \left(\sin \frac{7}{6} x - \sin \frac{1}{6} x \right) dx \\
&= \frac{1}{4} \int \left(\sin \frac{11}{6} x - \sin \frac{5}{6} x \right) dx \\
&= -\frac{3}{14} \cos \frac{7}{6} x + \frac{3}{2} \cos \frac{x}{6} + \frac{3}{22} \cos \frac{11}{6} x \\
&\quad - \frac{3}{10} \cos \frac{5}{6} x + C.
\end{aligned}$$

2016. $\int \sin x \sin(x+a) \sin(x+b) dx.$

$$\begin{aligned}
\text{解} \quad & \int \sin x \sin(x+a) \sin(x+b) dx \\
&= \frac{1}{2} \int \sin x [\cos(a-b) - \cos(2x+a+b)] dx \\
&= -\frac{1}{2} \cos x \cos(a-b) - \frac{1}{4} \int [\sin(3x+a+b) \\
&\quad - \sin(x+a+b)] dx \\
&= -\frac{1}{2} \cos x \cos(a-b) + \frac{1}{12} \cos(3x+a+b) \\
&\quad - \frac{1}{4} \cos(x+a+b) + C.
\end{aligned}$$

$$2017. \int \cos^2 ax \cos^2 bx dx.$$

$$\begin{aligned} \text{解} \quad & \int \cos^2 ax \cos^2 bx dx = \int (\cos ax \cos bx)^2 dx \\ & = \frac{1}{4} \int [\cos(a-b)x + \cos(a+b)x]^2 dx \\ & = \frac{1}{4} \int [\cos^2(a-b)x + \cos^2(a+b)x \\ & \quad + 2\cos(a-b)x\cos(a+b)x] dx. \\ & = \frac{1}{8} \int [2 + \cos 2(a+b)x + \cos 2(a-b)x] dx \\ & \quad + \frac{1}{4} \int (\cos 2ax + \cos 2bx) dx \\ & = \frac{x}{4} + \frac{\sin 2(a+b)x}{16(a+b)} + \frac{\sin 2(a-b)x}{16(a-b)} \\ & \quad + \frac{1}{8a} \sin 2ax + \frac{1}{8b} \sin 2bx + C. \end{aligned}$$

$$2018. \int \sin^3 2x \cos^2 3x dx.$$

解 先利用三角公式化简 $\sin^3 2x \cos^2 3x$, 得

$$\begin{aligned} \sin^3 2x \cos^2 3x &= -\frac{1}{16} \sin 12x + \frac{3}{16} \sin 8x \\ &\quad - \frac{1}{8} \sin 6x - \frac{3}{16} \sin 4x + \frac{3}{8} \sin 2x, \end{aligned}$$

于是

$$\int \sin^3 2x \cos^2 3x dx$$

$$= \frac{1}{192} \cos 12x - \frac{3}{128} \cos 8x + \frac{1}{48} \cos 6x \\ + \frac{3}{64} \cos 4x - \frac{3}{16} \cos 2x + C.$$

运用恒等式

$$\sin(\alpha - \beta) = \sin[(x + \alpha) - (x + \beta)]$$

$$\text{及 } \cos(\alpha - \beta) = \cos[(x + \alpha) - (x + \beta)]$$

来计算积分.

求积分:

$$2019. \int \frac{dx}{\sin(x+a)\sin(x+b)}.$$

$$\text{解 } \int \frac{dx}{\sin(x+a)\sin(x+b)}$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(a+x)-(x+b)]}{\sin(x+a)\sin(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\cos(x+b)}{\sin(x+b)} - \frac{\cos(x+a)}{\sin(x+a)} \right] dx$$

$$= \frac{1}{\sin(a-b)} \ln \left| \frac{\sin(x+b)}{\sin(x+a)} \right| + C,$$

其中设 $\sin(a-b) \neq 0$.

$$2020. \int \frac{dx}{\sin(x+a)\cos(x+b)}.$$

$$\text{解 } \int \frac{dx}{\sin(x+a)\cos(x+b)}$$

$$\begin{aligned}
&= \frac{1}{\cos(a-b)} \int \frac{\cos[(x+a)-(x+b)]}{\sin(x+a)\cos(x+b)} dx \\
&= \frac{1}{\cos(a-b)} \int \left[\frac{\cos(x+a)}{\sin(x+a)} + \frac{\sin(x+b)}{\cos(x+b)} \right] dx \\
&= \frac{1}{\cos(a-b)} \ln \left| \frac{\sin(x+a)}{\cos(x+b)} \right| + C,
\end{aligned}$$

其中设 $\cos(a-b) \neq 0$.

2021. $\int \frac{dx}{\cos(x+a)\cos(x+b)}$.

解 $\int \frac{dx}{\cos(x+a)\cos(x+b)}$

$$\begin{aligned}
&= \frac{1}{\sin(a-b)} \int \frac{\sin[(x+a)-(x+b)] dx}{\cos(x+a)\cos(x+b)} \\
&= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)} \right] dx \\
&= \frac{1}{\sin(a-b)} \ln \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C,
\end{aligned}$$

其中设 $\sin(a-b) \neq 0$ *).

*). 当 $a-b=2k\pi$ ($k=0, \pm 1, \pm 2, \dots$) 时, 是更简单的积分, 2019题及2020题与本题类似, 解法从略.

2022. $\int \frac{dx}{\sin x - \sin a}$.

解 $\int \frac{dx}{\sin x - \sin a} = \frac{1}{\cos a} \int \frac{\cos\left(\frac{x+a}{2} - \frac{x-a}{2}\right)}{\sin x - \sin a} dx$

$$\begin{aligned}
&= \frac{1}{\cos a} \int \frac{\cos \frac{x+a}{2} \cos \frac{x-a}{2} + \sin \frac{x+a}{2} \sin \frac{x-a}{2}}{2 \cos \frac{x+a}{2} \sin \frac{x-a}{2}} dx \\
&= \frac{1}{2 \cos a} \int \left(\frac{\cos \frac{x-a}{2}}{\sin \frac{x-a}{2}} + \frac{\sin \frac{x+a}{2}}{\cos \frac{x+a}{2}} \right) dx \\
&= \frac{1}{\cos a} \ln \left| \frac{\sin \frac{x-a}{2}}{\cos \frac{x+a}{2}} \right| + C,
\end{aligned}$$

其中设 $\cos a \neq 0$.

2023. $\int \frac{dx}{\cos x + \cos a}$.

$$\begin{aligned}
\text{解} \quad \int \frac{dx}{\cos x + \cos a} &= \int \frac{d\left(x + \frac{\pi}{2}\right)}{\sin\left(x + \frac{\pi}{2}\right) - \sin\left(a + \frac{3}{2}\pi\right)} \\
&= \frac{1}{\cos\left(a + \frac{3}{2}\pi\right)} \ln \left| \frac{\sin \frac{x-a-\pi}{2}}{\cos \frac{x+a+2\pi}{2}} \right|^{*}) + C \\
&= \frac{1}{\sin a} \ln \left| \frac{\cos \frac{x-a}{2}}{\cos \frac{x+a}{2}} \right| + C,
\end{aligned}$$

其中设 $\sin a \neq 0$.

*) 利用2022题的结果.

$$2024. \int \operatorname{tg} x \operatorname{tg}(x+a) dx.$$

$$\begin{aligned} \text{解} \quad & \int \operatorname{tg} x \operatorname{tg}(x+a) dx = \int \frac{\sin x \sin(x+a)}{\cos x \cos(x+a)} dx \\ & = \int \frac{\cos x \cos(x+a) + \sin x \sin(x+a) - \cos x \cos(x+a)}{\cos x \cos(x+a)} dx \\ & = \int \frac{\cos a - \cos x \cos(x+a)}{\cos x \cos(x+a)} dx \\ & = -x + \cos a \cdot \int \frac{dx}{\cos(x+a) \cos x} \\ & = -x + \operatorname{ctg} a \cdot \ln \left| \frac{\cos x}{\cos(x-a)} \right|^{*} + C, \end{aligned}$$

其中设 $\sin a \neq 0$.

*) 利用2021题的结果.

形如

$$\int R(\sin x, \cos x) dx$$

(式中 R 为有理函数) 的积分的一般情形可利用代换 $\operatorname{tg} \frac{x}{2} = t$ 化为有理函数的积分.

(a) 若等式

$$R(-\sin x, \cos x) \equiv -R(\sin x, \cos x),$$

或 $R(\sin x, -\cos x) \equiv -R(\sin x, \cos x)$

成立, 则最好利用代换 $\cos x = t$ 或对应的 $\sin x = t$.

(b) 若等式

$$R(-\sin x, -\cos x) \equiv R(\sin x, \cos x)$$

成立, 则最好利用代换 $\operatorname{tg} x = t$.

求积分:

$$2025. \int \frac{dx}{2\sin x - \cos x + 5}.$$

解 设 $t = \operatorname{tg} \frac{x}{2}$, 则 $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$,

$$dx = \frac{2dt}{1+t^2}.$$

于是,

$$\begin{aligned} \int \frac{dx}{2\sin x - \cos x + 5} &= \int \frac{dt}{3t^2 + 2t + 2} \\ &= \frac{1}{\sqrt{5}} \operatorname{arc} \operatorname{tg} \left(\frac{3t + 1}{\sqrt{5}} \right) + C \\ &= \frac{1}{\sqrt{5}} \operatorname{arc} \operatorname{tg} \left(\frac{3\operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} \right) + C. \end{aligned}$$

$$2026. \int \frac{dx}{(2 + \cos x)\sin x}.$$

解 设 $t = \operatorname{tg} \frac{x}{2}$. 同2025题, 得

$$\begin{aligned} \int \frac{dx}{(2 + \cos x)\sin x} &= \int \frac{1+t^2}{t(3+t^2)} dt \\ &= \int \left[\frac{1}{3t} + \frac{2t}{3(3+t^2)} \right] dt \\ &= \frac{1}{3} \ln |t(3+t^2)| + C_1 \end{aligned}$$

$$= \frac{1}{6} \ln \frac{(1 - \cos x)(2 + \cos x)^2}{(1 + \cos x)^3} + C. \quad *)$$

*) 由于

$$\begin{aligned} t(3+t^2) &= \operatorname{tg} \frac{x}{2} \left(2 + \sec^2 \frac{x}{2} \right) \\ &= \frac{\sin \frac{x}{2}}{\cos^3 \frac{x}{2}} \left(1 + 2 \cos^2 \frac{x}{2} \right) \\ &= \frac{\left(\frac{1 - \cos x}{2} \right)^{\frac{1}{2}}}{\left(\frac{1 + \cos x}{2} \right)^{\frac{3}{2}}} (\cos x + 2) \\ &= 2 \left[\frac{(1 - \cos x)(\cos x + 2)^2}{(1 + \cos x)^3} \right]^{\frac{1}{2}}, \end{aligned}$$

因而

$$\ln |t(3+t^2)| = \ln 2 + \frac{1}{2} \ln \frac{(1 - \cos x)(2 + \cos x)^2}{(1 + \cos x)^3}.$$

2027. $\int \frac{\sin^2 x}{\sin x + 2 \cos x} dx.$

解 设 $\operatorname{tg} \frac{x}{2} = t$, 同2025题, 得

$$\begin{aligned} \int \frac{\sin^2 x}{\sin x + 2 \cos x} dx &= 4 \int \frac{t^2 dt}{(1+t^2)^2(1+t-t^2)} \\ &= \frac{4}{5} \int \left[\frac{1}{1+t^2} + \frac{-2+t}{(1+t^2)^2} + \frac{1}{1+t-t^2} \right] dt \end{aligned}$$

$$\begin{aligned}
&= \frac{4}{5} \int \frac{dt}{1+t^2} - \frac{8}{5} \int \frac{dt}{(1+t^2)^2} + \frac{2}{5} \int \frac{2tdt}{(1+t^2)^2} \\
&+ \frac{4}{5} \int \frac{d\left(t - \frac{1}{2}\right)}{\frac{5}{4} - \left(t - \frac{1}{2}\right)^2} \\
&= \frac{4}{5} \operatorname{arctg} t - \frac{8}{5} \left[\frac{t}{2(1+t^2)} + \frac{1}{2} \operatorname{arc} \operatorname{tg} t \right]^{*}) \\
&- \frac{2}{5} \cdot \frac{1}{1+t^2} + \frac{4}{5\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}}{2} + t - \frac{1}{2}}{\frac{\sqrt{5}}{2} - \left(t - \frac{1}{2}\right)} \right| + C_1 \\
&= -\frac{2}{5} \cdot \frac{1+2t}{1+t^2} + \frac{4}{5\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}-1}{2} + t}{\frac{\sqrt{5}+1}{2} - t} \right| + C_1 \\
&= -\frac{1}{5} (\cos x + 2\sin x)^{**}) \\
&+ \frac{4}{5\sqrt{5}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\operatorname{arctg} 2}{2} \right) \right|^{***}) + C.
\end{aligned}$$

*) 利用1817题的结果。

$$***) \quad -\frac{2}{5} \cdot \frac{1+2t}{1+t^2} = -\frac{2}{5} \cdot \frac{1+2\operatorname{tg} \frac{x}{2}}{\sec^2 \frac{x}{2}}$$

$$= -\frac{2}{5} \cdot \frac{1+2 \cdot \frac{\sin x}{1+\cos x}}{1+\cos x} = -\frac{1}{5} (\cos x + 2\sin x) - \frac{1}{5}.$$

$$\begin{aligned}
 \text{***)} \quad \ln \left| \frac{\frac{\sqrt{5}-1}{2} + t}{\frac{\sqrt{5}+1}{2} - t} \right| &= \ln \left| \frac{\operatorname{tg}\left(\frac{\operatorname{arctg} 2}{2}\right) + \operatorname{tg} \frac{x}{2}}{\operatorname{ctg}\left(\frac{\operatorname{arctg} 2}{2}\right) - \operatorname{tg} \frac{x}{2}} \right| \\
 &= \ln \left| \frac{\operatorname{tg}\left(\frac{\operatorname{arctg} 2}{2}\right) + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg}\left(\frac{\operatorname{arctg} 2}{2}\right) \cdot \operatorname{tg} \frac{x}{2}} \right| + \ln \frac{1}{\operatorname{ctg}\left(\frac{\operatorname{arctg} 2}{2}\right)} \\
 &= \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\operatorname{arctg} 2}{2}\right) \right| - \ln \left[\operatorname{ctg}\left(\frac{\operatorname{arctg} 2}{2}\right) \right].
 \end{aligned}$$

2028. $\int \frac{dx}{1 + \varepsilon \cos x}$;

(a) $0 < \varepsilon < 1$; (b) $\varepsilon > 1$.

解 设 $t = \operatorname{tg} \frac{x}{2}$. 同2025题, 得

$$\int \frac{dx}{1 + \varepsilon \cos x} = 2 \int \frac{dt}{(1 + \varepsilon) + (1 - \varepsilon)t^2} = I.$$

(a) $0 < \varepsilon < 1$,

$$I = \frac{2}{1 + \varepsilon} \int \frac{dt}{1 + \left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)t^2}$$

$$= \frac{2}{\sqrt{1 - \varepsilon^2}} \operatorname{arctg} \left(t \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} \right) + C$$

$$= \frac{2}{\sqrt{1 - \varepsilon^2}} \operatorname{arctg} \left(\sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} \operatorname{tg} \frac{x}{2} \right) + C;$$

$$(6) \quad e > 1,$$

$$I = \frac{2}{e-1} \int \frac{dt}{\left(\frac{e+1}{e-1}\right) - t^2}$$

$$= \frac{1}{\sqrt{e^2-1}} \ln \left| \frac{\sqrt{e+1} + \sqrt{e-1}t}{\sqrt{e+1} - \sqrt{e-1}t} \right| + C$$

$$= \frac{1}{\sqrt{e^2-1}} \ln \left| \frac{e + \cos x + \sqrt{e^2-1} \sin x}{1 + e \cos x} \right|^{*}) + C.$$

$$*) \frac{\sqrt{e+1} + t\sqrt{e-1}}{\sqrt{e+1} - t\sqrt{e-1}} = \frac{e+1 + 2t\sqrt{e^2-1} + (e-1)t^2}{(e+1) - (e-1)t^2}$$

$$= \frac{e(1+t^2) + (1-t^2) + 2\sqrt{e^2-1}t}{e(1-t^2) + (1+t^2)}$$

$$= \frac{e(1+t^2) + (1+t^2)\cos x + 2t\sqrt{e^2-1}}{e(1+t^2)\cos x + (1+t^2)}$$

$$= \frac{e + \cos x + \sqrt{e^2-1} \cdot \frac{2t}{1+t^2}}{e \cos x + 1}$$

$$= \frac{e + \cos x + \sqrt{e^2-1} \sin x}{e \cos x + 1}$$

$$2029. \int \frac{\sin^2 x}{1 + \sin^2 x} dx.$$

$$\text{解} \quad \int \frac{\sin^2 x}{1 + \sin^2 x} dx = \int \left(1 - \frac{1}{1 + \sin^2 x} \right) dx$$

$$= x - \int \frac{d(\operatorname{tg} x)}{\sec^2 x + \operatorname{tg}^2 x} = x - \int \frac{d(\operatorname{tg} x)}{1 + 2\operatorname{tg}^2 x}$$

$$= x - \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg}(\sqrt{2} \operatorname{tg} x) + C.$$

$$2030. \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} &= \int \frac{d\left(\frac{\sin x}{\cos x}\right)}{a^2 \operatorname{tg}^2 x + b^2} \\ &= \frac{1}{ab} \operatorname{arc} \operatorname{tg}\left(\frac{a \operatorname{tg} x}{b}\right) + C, \end{aligned}$$

其中设 $ab \neq 0$.

$$2031. \int \frac{\cos^2 x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}.$$

$$\begin{aligned} \text{解} \quad \int \frac{\cos^2 x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} &= \frac{1}{a} \int \frac{d(a \operatorname{tg} x)}{(a^2 \operatorname{tg}^2 x + b^2)^2} \\ &= \frac{\operatorname{tg} x}{2b^2(a^2 \operatorname{tg}^2 x + b^2)} + \frac{1}{2ab^3} \operatorname{arc} \operatorname{tg}\left(\frac{a \operatorname{tg} x}{b}\right)^* + C, \end{aligned}$$

其中设 $ab \neq 0$.

*.) 利用1921题的结果.

$$2032. \int \frac{\sin x \cos x}{\sin x + \cos x} dx.$$

$$\text{解} \quad \int \frac{\sin x \cos x}{\sin x + \cos x} dx = \int \frac{\sin^2\left(x + \frac{\pi}{4}\right) - \frac{1}{2}}{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)} dx$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \int \sin\left(x + \frac{\pi}{4}\right) dx - \frac{1}{2\sqrt{2}} \int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} \\
&= -\frac{1}{\sqrt{2}} \cos\left(x + \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}} \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{8}\right) \right| + C \\
&= \frac{1}{2} (\sin x - \cos x) - \frac{1}{2\sqrt{2}} \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{8}\right) \right| + C.
\end{aligned}$$

2033. $\int \frac{dx}{(a \sin x + b \cos x)^2}$.

解 $\int \frac{dx}{(a \sin x + b \cos x)^2} = \frac{1}{a} \int \frac{d(atg x + b)}{(atg x + b)^2}$

$$= -\frac{1}{atg x + b} + C = -\frac{\cos x}{a(a \sin x + b \cos x)} + C.$$

2034. $\int \frac{\sin x dx}{\sin^3 x + \cos^3 x}$.

解 $\int \frac{\sin x dx}{\sin^3 x + \cos^3 x} = \int \frac{\sin x dx}{(\sin x + \cos x)(1 - \sin x \cos x)}$

$$= \frac{1}{2} \int \frac{(\sin x - \cos x) dx}{(\sin x + \cos x)(1 - \sin x \cos x)}$$

$$+ \frac{1}{2} \int \frac{dx}{1 - \sin x \cos x}$$

$$= \frac{1}{3} \int \frac{-(\cos x - \sin x) dx}{\sin x + \cos x}$$

$$+ \frac{1}{6} \int \frac{\sin^2 x - \cos^2 x}{1 - \sin x \cos x} dx + \frac{1}{2} \int \frac{dx}{1 - \sin x \cos x}$$

$$\begin{aligned}
&= -\frac{1}{3} \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} + \frac{1}{6} \int \frac{d(1 - \sin x \cos x)}{1 - \sin x \cos x} \\
&= -\frac{1}{\sqrt{3}} \int d\left(\operatorname{arc} \operatorname{tg} \frac{2 \cos x - \sin x}{\sqrt{3} \sin x}\right) \\
&= -\frac{1}{6} \ln \frac{(\sin x + \cos x)^2}{1 - \sin x \cos x} \\
&= -\frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2 \cos x - \sin x}{\sqrt{3} \sin x} \right) + C.
\end{aligned}$$

2035. $\int \frac{dx}{\sin^4 x + \cos^4 x}$

解 $\int \frac{dx}{\sin^4 x + \cos^4 x} = \int \frac{2dx}{2 - \sin^2 2x}$

$$\begin{aligned}
&= \int \frac{d(\operatorname{tg} 2x)}{2 \sec^2 2x - \operatorname{tg}^2 2x} \\
&= \int \frac{d(\operatorname{tg} 2x)}{2 + \operatorname{tg}^2 2x} = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} 2x}{\sqrt{2}} \right) + C.
\end{aligned}$$

2036. $\int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x} dx$

解 $\int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x} dx = \int \frac{2 \sin^2 2x dx}{\sin^4 2x - 8 \sin^2 2x + 8}$

$$\begin{aligned}
&= \int \frac{\operatorname{tg}^2 2x d(\operatorname{tg} 2x)}{\operatorname{tg}^4 2x - 8 \operatorname{tg}^2 2x \sec^2 2x + 8 \sec^4 2x} \\
&= \int \frac{\operatorname{tg}^2 2x d(\operatorname{tg} 2x)}{\operatorname{tg}^4 2x + 8 \operatorname{tg}^2 2x + 8}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{4} (2 + \sqrt{2}) \int \frac{d(\operatorname{tg} 2x)}{\operatorname{tg}^2 2x + 4 + 2\sqrt{2}} \\
&- \frac{\sqrt{2}}{4} (2 - \sqrt{2}) \int \frac{d(\operatorname{tg} 2x)}{\operatorname{tg}^2 2x + 4 - \sqrt{2}} \\
&= \frac{1}{4} \left[\sqrt{2 + \sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} 2x}{\sqrt{4 + 2\sqrt{2}}} \right) \right. \\
&\left. - \sqrt{2 - \sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} 2x}{\sqrt{4 - 2\sqrt{2}}} \right) \right] + C.
\end{aligned}$$

2037. $\int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx.$

解 $\int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx = - \int \frac{\cos 2x}{1 - \frac{1}{2} \sin^2 2x} dx$

$$= - \frac{1}{2\sqrt{2}} \int \left(\frac{2\cos 2x}{\sqrt{2} - \sin 2x} + \frac{2\cos 2x}{\sqrt{2} + \sin 2x} \right) dx$$

$$= \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2} - \sin 2x}{\sqrt{2} + \sin 2x} + C.$$

2038. $\int \frac{\sin x \cos x}{1 + \sin^4 x} dx.$

解 $\int \frac{\sin x \cos x}{1 + \sin^4 x} dx = \int \frac{\operatorname{tg} x \sec^2 x}{\sec^4 x + \operatorname{tg}^4 x} dx$

$$= \frac{1}{2} \int \frac{d(\operatorname{tg}^2 x)}{2\operatorname{tg}^4 x + 2\operatorname{tg}^2 x + 1} = \frac{1}{2} \operatorname{arctg}(1 + 2\operatorname{tg}^2 x) + C.$$

2039. $\int \frac{dx}{\sin^6 x + \cos^6 x}.$

$$\begin{aligned}
 \text{解} \quad & \int \frac{dx}{\sin^4 x + \cos^4 x} = \int \frac{dx}{1 - 3\sin^2 x \cos^2 x} \\
 & = \int \frac{dx}{1 - \frac{3}{4}\sin^2 2x} = \int \frac{2d(\operatorname{tg} 2x)}{4\sec^2 2x - 3\operatorname{tg}^2 2x} \\
 & = \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} 2x}{2} \right) + C.
 \end{aligned}$$

$$2040. \int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2}$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2} = \int \frac{\frac{1}{\cos^4 x} dx}{(\operatorname{tg}^2 x + 2)^2} \\
 & = \int \frac{\sec^2 x d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2} \\
 & = \int \frac{\operatorname{tg}^2 x}{(\operatorname{tg}^2 x + 2)^2} d(\operatorname{tg} x) + \int \frac{d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2} \\
 & = \int \frac{(\operatorname{tg}^2 x + 2) - 2}{(\operatorname{tg}^2 x + 2)^2} d(\operatorname{tg} x) + \int \frac{d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2} \\
 & = \int \frac{d(\operatorname{tg} x)}{\operatorname{tg}^2 x + 2} - \int \frac{d(\operatorname{tg} x)}{(\operatorname{tg}^2 x + 2)^2} \\
 & = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right) - \frac{\operatorname{tg} x}{4(\operatorname{tg}^2 x + 2)} \\
 & \quad - \frac{1}{4\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right)^* + C
 \end{aligned}$$

$$= \frac{3}{4\sqrt{2}} \operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right) - \frac{\operatorname{tg}x}{4(\operatorname{tg}^2x+2)} + C.$$

*) 利用1817题的结果.

2041. 求积分

$$\int \frac{dx}{a\sin x + b\cos x}$$

先化分母为对数的形状.

$$\begin{aligned} \text{解 } \int \frac{dx}{a\sin x + b\cos x} &= \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\sin(x+\varphi)} \\ &= \frac{1}{\sqrt{a^2+b^2}} \ln \left| \operatorname{tg}\left(\frac{x+\varphi}{2}\right) \right| + C, \end{aligned}$$

$$\text{其中 } \cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}},$$

并设 $a^2+b^2 \neq 0$.

2042. 证明

$$\int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} dx = Ax + B \ln |a \sin x + b \cos x| + C,$$

式中 A, B, C 为常数.

$$\begin{aligned} \text{证 } a_1 \sin x + b_1 \cos x &= A(a \sin x + b \cos x) \\ &+ B(a \cos x - b \sin x), \end{aligned}$$

$$\text{式中 } A = \frac{aa_1 + bb_1}{a^2 + b^2}, \quad B = \frac{ab_1 - a_1b}{a^2 + b^2}, \quad a^2 + b^2 \neq 0.$$

于是

$$\int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} dx = A \int dx$$

$$+ B \int \frac{d(a \sin x + b \cos x)}{a \sin x + b \cos x}$$

$$= Ax + B \ln |a \sin x + b \cos x| + C.$$

求积分:

$$2043. \int \frac{\sin x - \cos x}{\sin x + 2 \cos x} dx.$$

解 此为2042题的特例, 这里

$$a_1 = 1, b_1 = -1, a = 1, b = 2;$$

$$A = \frac{aa_1 + bb_1}{a^2 + b^2} = \frac{1 - 2}{1 + 4} = -\frac{1}{5},$$

$$B = \frac{ab_1 - a_1b}{a^2 + b^2} = \frac{-1 - 2}{1 + 4} = -\frac{3}{5}.$$

代入得

$$\int \frac{\sin x - \cos x}{\sin x + 2 \cos x} dx = -\frac{x}{5}$$

$$- \frac{3}{5} \ln |\sin x + 2 \cos x| + C.$$

$$2044. \int \frac{dx}{3 + 5 \operatorname{tg} x}.$$

$$\text{解 } \int \frac{dx}{3 + 5 \operatorname{tg} x} = \int \frac{\cos x}{5 \sin x + 3 \cos x} dx.$$

此为2042题的特例, 这里

$$a_1 = 0, b_1 = 1, a = 5, b = 3;$$

$$A = \frac{3}{34}, B = \frac{5}{34}.$$

代入得

$$\int \frac{dx}{3+5\operatorname{tg}x} = \frac{3}{34}x + \frac{5}{34}\ln|5\sin x + 3\cos x| + C.$$

2045. $\int \frac{a_1 \sin x + b_1 \cos x}{(a \sin x + b \cos x)^2} dx.$

解 仿2042題, $\int \frac{a_1 \sin x + b_1 \cos x}{(a \sin x + b \cos x)^2} dx$

$$= A \int \frac{a \sin x + b \cos x}{(a \sin x + b \cos x)^2} dx$$

$$+ B \int \frac{a \cos x - b \sin x}{(a \sin x + b \cos x)^2} dx$$

$$= A \int \frac{dx}{a \sin x + b \cos x} + B \int \frac{d(a \sin x + b \cos x)}{(a \sin x + b \cos x)^2} dx$$

$$= \frac{A}{\sqrt{a^2 + b^2}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\varphi}{2} \right) \right|^{*}) - \frac{B}{a \sin x + b \cos x} + C$$

$$= \frac{aa_1 + bb_1}{(a^2 + b^2)^{\frac{3}{2}}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\varphi}{2} \right) \right|$$

$$- \frac{ab_1 - a_1 b}{(a^2 + b^2)(a \sin x + b \cos x)} + C,$$

式中 $A = \frac{aa_1 + bb_1}{a^2 + b^2}, B = \frac{ab_1 - a_1 b}{a^2 + b^2},$

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}},$$

$a^2+b^2 \neq 0$ (显然按题意 a, b 不同时为零)。

*) 利用2041题的结果。

2046. 证明:

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x + c_1}{a \sin x + b \cos x + c} dx \\ &= Ax + B \ln |a \sin x + b \cos x + c| \\ &+ C \int \frac{dx}{a \sin x + b \cos x + c}, \end{aligned}$$

式中 A, B, C 都是常系数。

证 按题意 a, b 不同时为零。设

$$\begin{aligned} a_1 \sin x + b_1 \cos x + c_1 &\equiv A(a \sin x + b \cos x + c) \\ &+ B(a \cos x - b \sin x) + C, \end{aligned}$$

比较等式两端同类项的系数, 则有

$$\begin{aligned} A &= \frac{aa_1 + bb_1}{a^2 + b^2}, \quad B = \frac{ab_1 - a_1b}{a^2 + b^2}, \\ C &= \frac{a(ac_1 - a_1c) + b(bc_1 - b_1c)}{a^2 + b^2}. \end{aligned}$$

代入得

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x + c_1}{a \sin x + b \cos x + c} dx \\ &= A \int dx + B \int \frac{d(a \sin x + b \cos x + c)}{a \sin x + b \cos x + c} \end{aligned}$$

$$\begin{aligned}
& + C \int \frac{dx}{a \sin x + b \cos x + c} \\
& = Ax + B \ln |a \sin x + b \cos x + c| \\
& + C \int \frac{dx}{a \sin x + b \cos x + c}.
\end{aligned}$$

求积分:

$$2047. \int \frac{\sin x + 2 \cos x - 3}{\sin x - 2 \cos x + 3} dx.$$

解 此为2046题之特例, 这里

$$a_1 = 1, b_1 = 2, c_1 = -3, a = 1, b = -2, c = 3,$$

$$A = \frac{aa_1 + bb_1}{a^2 + b^2} = \frac{1 - 4}{1 + 4} = -\frac{3}{5},$$

$$B = \frac{ab_1 - a_1b}{a^2 + b^2} = \frac{2 + 2}{1 + 4} = \frac{4}{5},$$

$$\begin{aligned}
C &= \frac{a(ac_1 - a_1c) + b(bc_1 - b_1c)}{a^2 + b^2} \\
&= \frac{(-3 - 3) + (-2)(6 - 6)}{1 + 4} = -\frac{6}{5}.
\end{aligned}$$

代入得

$$\begin{aligned}
& \int \frac{\sin x + 2 \cos x - 3}{\sin x - 2 \cos x + 3} dx \\
& = -\frac{3}{5}x + \frac{4}{5} \ln |\sin x - 2 \cos x + 3| \\
& \quad - \frac{6}{5} \int \frac{dx}{\sin x - 2 \cos x + 3}.
\end{aligned}$$

$$= -\frac{3}{5}x + \frac{4}{5}\ln|\sin x - 2\cos x + 3|$$

$$- \frac{6}{5} \operatorname{arc} \operatorname{tg} \frac{1 + 5 \operatorname{tg} \frac{x}{2}}{2} + C.$$

*) 设 $t = \operatorname{tg} \frac{x}{2}$, 积分即得所求式子.

$$2048. \int \frac{\sin x dx}{\sqrt{2} + \sin x + \cos x}.$$

解 此为2046题之特例, 这里

$$a_1 = 1, b_1 = 0, c_1 = 0, a = 1, b = 1, c = \sqrt{2};$$

$$A = \frac{1}{2}, B = -\frac{1}{2}, C = -\frac{1}{\sqrt{2}}.$$

代入得

$$\begin{aligned} & \int \frac{\sin x dx}{\sqrt{2} + \sin x + \cos x} \\ &= \frac{1}{2}x - \frac{1}{2}\ln|\sqrt{2} + \sin x + \cos x| \\ & \quad - \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{2} + \sin x + \cos x} \\ &= \frac{1}{2}x - \frac{1}{2}\ln|\sqrt{2} + \sin x + \cos x| \\ & \quad - \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{2} + \sqrt{2} \cos(x - \frac{\pi}{4})} \end{aligned}$$

$$= \frac{1}{2}x - \frac{1}{2} \ln |\sqrt{2} + \sin x + \cos x|$$

$$- \frac{1}{2} \int \frac{dx}{2 \cos^2 \left(\frac{x}{2} - \frac{\pi}{8} \right)}$$

$$= \frac{1}{2}x - \frac{1}{2} \ln |\sqrt{2} + \sin x + \cos x|$$

$$- \frac{1}{2} \operatorname{tg} \left(\frac{x}{2} - \frac{\pi}{8} \right) + C.$$

2049. $\int \frac{2 \sin x + \cos x}{3 \sin x + 4 \cos x - 2} dx.$

解 本题也是2046题之特例, 这里

$$a_1 = 2, b_1 = 1, c_1 = 0, a = 3, b = 4, c = -2;$$

$$A = \frac{2}{5}, B = -\frac{1}{5}, C = \frac{4}{5}.$$

代入得

$$\int \frac{2 \sin x + \cos x}{3 \sin x + 4 \cos x - 2} dx$$

$$= \frac{2}{5}x - \frac{1}{5} \ln |3 \sin x + 4 \cos x - 2|$$

$$+ \frac{4}{5} \int \frac{dx}{3 \sin x + 4 \cos x - 2}$$

$$= \frac{2}{5}x - \frac{1}{5} \ln |3 \sin x + 4 \cos x - 2|$$

$$+ \frac{4}{5 \sqrt{21}} \ln \left| \frac{\sqrt{7} + \sqrt{3} \left(2 \operatorname{tg} \frac{x}{2} - 1 \right)}{\sqrt{7} - \sqrt{3} \left(2 \operatorname{tg} \frac{x}{2} - 1 \right)} \right|^{(*)} + C.$$

*) 设 $t = \operatorname{tg} \frac{x}{2}$, 积分即得所求式子.

2050. 证明:

$$\int \frac{a_1 \sin^2 x + 2b_1 \sin x \cos x + c_1 \cos^2 x}{a \sin x + b \cos x} dx$$

$$= A \sin x + B \cos x + C \int \frac{dx}{a \sin x + b \cos x},$$

式中 A, B, C 都是常系数.

证 按题意 a, b 不同时为零. 设

$$a_1 \sin^2 x + 2b_1 \sin x \cos x + c_1 \cos^2 x$$

$$= A \cos x (a \sin x + b \cos x) - B \sin x (a \sin x + b \cos x) + C,$$

比较等式两端同类项的系数, 则有

$$aA - bB = 2b_1, \quad C - aB = a_1, \quad C + bA = c_1,$$

从而

$$A = \frac{bc_1 - a_1 b + 2ab_1}{a^2 + b^2}, \quad B = \frac{ac_1 - aa_1 - 2bb_1}{a^2 + b^2},$$

$$C = \frac{a_1 b^2 + a^2 c_1 - 2abb_1}{a^2 + b^2}.$$

代入得

$$\int \frac{a_1 \sin^2 x + 2b_1 \sin x \cos x + c_1 \cos^2 x}{a \sin x + b \cos x} dx$$

$$= A \int \cos x dx - B \int \sin x dx + C \int \frac{dx}{a \sin x + b \cos x}$$

$$= A \sin x + B \cos x + C \int \frac{dx}{a \sin x + b \cos x}.$$

求积分:

$$2051. \int \frac{\sin^2 x - 4\sin x \cos x + 3\cos^2 x}{\sin x + \cos x} dx.$$

解 此为2050题之特例, 这里

$$a_1 = 1, b_1 = -2, c_1 = 3, a = 1, b = 1;$$

$$A = \frac{bc_1 - a_1 b + 2ab_1}{a^2 + b^2} = \frac{3 - 1 - 4}{1 + 1} = -1,$$

$$B = \frac{ac_1 - aa_1 - 2bb_1}{a^2 + b^2} = \frac{3 - 1 + 4}{1 + 1} = 3,$$

$$C = \frac{a_1 b^2 + a^2 c_1 - 2abb_1}{a^2 + b^2} = \frac{1 + 3 + 4}{1 + 1} = 4.$$

代入得

$$\begin{aligned} & \int \frac{\sin^2 x - 4\sin x \cos x + 3\cos^2 x}{\sin x + \cos x} dx \\ &= -\sin x + 3\cos x + 4 \int \frac{dx}{\sin x + \cos x} \\ &= -\sin x + 3\cos x + \frac{4}{\sqrt{2}} \int \frac{dx}{\sin(x - \frac{\pi}{4})} \\ &= -\sin x + 3\cos x + 2\sqrt{2} \ln \left| \operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{8}\right) \right| + C. \end{aligned}$$

$$2052. \int \frac{\sin^2 x - \sin x \cos x + 2\cos^2 x}{\sin x + 2\cos x} dx.$$

解 本题也是2050题的特例, 这里

$$a_1 = 1, b_1 = -\frac{1}{2}, c_1 = 2, a = 1, b = 2;$$

$$A = \frac{1}{5}, B = \frac{3}{5}, C = \frac{8}{5}.$$

代入得

$$\begin{aligned} & \int \frac{\sin^2 x - \sin x \cos x + 2 \cos^2 x}{\sin x + 2 \cos x} dx \\ &= \frac{1}{5} \sin x + \frac{3}{5} \cos x + \frac{8}{5} \int \frac{dx}{\sin x + 2 \cos x} \\ &= \frac{1}{5} (\sin x + 3 \cos x) \\ &+ \frac{8}{5 \sqrt{5}} \ln \left| \frac{\sqrt{5} + 2 \operatorname{tg} \frac{x}{2} - 1}{\sqrt{5} - 2 \operatorname{tg} \frac{x}{2} + 1} \right|^{*}) + C. \end{aligned}$$

*) 设 $t = \operatorname{tg} \frac{x}{2}$, 积分即得所求式子.

2053. 证明: 若 $(a-c)^2 + b^2 \neq 0$, 则

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x}{a \sin^2 x + 2b \sin x \cos x + c \cos^2 x} dx \\ &= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2}, \end{aligned}$$

式中 A, B 为未定系数, λ_1, λ_2 为下方程式的根

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0 \quad (\lambda_1 \neq \lambda_2)$$

及

$$u_i = (a - \lambda_i) \sin x + b \cos x, \quad k_i = \frac{1}{a - \lambda_i} \quad (i = 1, 2).$$

证 记

$$\begin{aligned}
& a^2 \sin^2 x + 2b \sin x \cos x + c \cos^2 x \\
&= (a - \lambda_i) \sin^2 x + 2b \sin x \cos x + (c - \lambda_i) \cos^2 x + \lambda_i \\
&= \frac{1}{a - \lambda_i} [(a - \lambda_i)^2 \sin^2 x + 2b(a - \lambda_i) \sin x \cos x \\
&\quad + (c - \lambda_i)(a - \lambda_i) \cos^2 x] + \lambda_i,
\end{aligned}$$

其中 $\lambda_i (i=1, 2)$ 为 $\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0$ 的根.

由假定 $(a - c)^2 + b^2 \neq 0$, 从而 $(a - c)^2 + 4b^2 \neq 0$, 因此 $\lambda_1 \neq \lambda_2$.

再设 $k_i = \frac{1}{a - \lambda_i} (i=1, 2)$ 及

$$u_i = (a - \lambda_i) \sin x + b \cos x.$$

由于 $(a - \lambda_i)(c - \lambda_i) - b^2 = 0$, 即 $b^2 = (a - \lambda_i)(c - \lambda_i)$. 于是,

$$\begin{aligned}
& a^2 \sin^2 x + 2b \sin x \cos x + c \cos^2 x \\
&= k_i [(a - \lambda_i)^2 \sin^2 x + 2b(a - \lambda_i) \sin x \cos x \\
&\quad + b^2 \cos^2 x] + \lambda_i = k_i [(a - \lambda_i) \sin x + b \cos x]^2 + \lambda_i \\
&= k_i u_i^2 + \lambda_i. \tag{1}
\end{aligned}$$

其次, 设

$$\begin{aligned}
a_1 \sin x + b_1 \cos x &= A[(a - \lambda_1) \cos x - b \sin x] \\
&\quad + B[(a - \lambda_2) \cos x - b \sin x], \tag{2}
\end{aligned}$$

比较等式两端同类项的系数, 则有

$$-b(A + B) = a_1,$$

$$A(a - \lambda_1) + B(a - \lambda_2) = b_1,$$

$$A = -\frac{a_1(\lambda_1 - \lambda_2) + b b_1 + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)},$$

$$B = \frac{bb_1 + a_1(a - \lambda_1)^*}{b(\lambda_1 - \lambda_2)}$$

由 (1) 式及 (2) 式即得

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x}{a^2 \sin^2 x + 2b \sin x \cos x + c \cos^2 x} dx \\ &= A \int \frac{(a - \lambda_1) \cos x - b \sin x}{k_1 [(a - \lambda_1) \sin x + b \cos x]^2 + \lambda_1} dx \\ &+ B \int \frac{(a - \lambda_2) \cos x - b \sin x}{k_2 [(a - \lambda_2) \sin x + b \cos x]^2 + \lambda_2} dx \\ &= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2} \end{aligned}$$

- * 按题意, $b \neq 0$. 因若 $b = 0$, 则 $\lambda_1 = a$, $\lambda_2 = c$, 从而 k_1 无意义. 不过, 当 $b = 0$ 时, 仍能化为所要求的类似形式. 事实上, 当 $b = 0$ 时, $a \neq c$,

我们有

$$\begin{aligned} & \int \frac{a_1 \sin x + b_1 \cos x}{a \sin^2 x + 2b \sin x \cos x + c \cos^2 x} dx \\ &= \int \frac{a_1 \sin x + b_1 \cos x}{a \sin^2 x + c \cos^2 x} dx \\ &= a_1 \int \frac{\sin x}{a \sin^2 x + c \cos^2 x} dx + b_1 \int \frac{\cos x}{a \sin^2 x + c \cos^2 x} dx \\ &= a_1 \int \frac{d(\cos x)}{(c - a) \cos^2 x + a} + b_1 \int \frac{d(\sin x)}{(a - c) \sin^2 x + c} \end{aligned}$$

$$= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2},$$

式中 $A = -a_1, B = b_1, k_1 = c - a, k_2 = a - c,$

$$u_1 = \cos x, u_2 = \sin x, \lambda_1 = a, \lambda_2 = c.$$

本题也可用下法另证: 命 $u_i = (a - \lambda_i) \sin x + b \cos x,$

$k_i = \frac{1}{a - \lambda_i} (i = 1, 2),$ 代入积分等式. 然后两边求导,

整理并比较系数, 便可知 λ_i 必为 $\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0$ 的根,

相应可求出系数, $A, B.$

求积分:

$$2054. \int \frac{2 \sin x - \cos x}{3 \sin^2 x + 4 \cos^2 x} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{2 \sin x - \cos x}{3 \sin^2 x + 4 \cos^2 x} dx \\ &= \int \frac{2 \sin x}{3 \sin^2 x + 4 \cos^2 x} dx - \int \frac{\cos x}{3 \sin^2 x + 4 \cos^2 x} dx \\ &= -2 \int \frac{d(\cos x)}{3 + \cos^2 x} - \int \frac{d(\sin x)}{4 - \sin^2 x} \\ &= -\frac{2}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \left(\frac{\cos x}{\sqrt{3}} \right) - \frac{1}{4} \ln \frac{2 + \sin x}{2 - \sin x} + C. \end{aligned}$$

$$2055. \int \frac{\sin x + \cos x}{2 \sin^2 x - 4 \sin x \cos x + 5 \cos^2 x} dx.$$

解 此为2053题之特例, 这里

$$a_1 = 1, b_1 = 1, a = 2, b = -2, c = 5.$$

由

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0,$$

求得 $\lambda_1 = 1, \lambda_2 = 6$, 从而

$$\begin{aligned} A &= -\frac{a_1(\lambda_1 - \lambda_2) + b_1 b + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)} \\ &= \frac{(1-6) - 2 + (2-1)}{-2(1-6)} = \frac{3}{5}, \end{aligned}$$

$$B = \frac{bb_1 + a_1(a - \lambda_1)}{b(\lambda_1 - \lambda_2)} = \frac{-2 + 1}{10} = -\frac{1}{10},$$

$$u_1 = (a - \lambda_1)\sin x + b\cos x = \sin x - 2\cos x,$$

$$u_2 = (a - \lambda_2)\sin x + b\cos x = -4\sin x - 2\cos x;$$

$$k_1 = \frac{1}{a - \lambda_1} = 1,$$

$$k_2 = \frac{1}{a - \lambda_2} = -\frac{1}{4}.$$

代入得

$$\begin{aligned} &\int \frac{\sin x + \cos x}{2\sin^2 x - 4\sin x \cos x + 5\cos^2 x} dx \\ &= \frac{3}{5} \int \frac{d(\sin x - 2\cos x)}{(\sin x - 2\cos x)^2 + 1} \\ &\quad + \frac{1}{10} \int \frac{d(4\sin x + 2\cos x)}{6 - \frac{1}{4}(4\sin x + 2\cos x)^2} \\ &= \frac{3}{5} \operatorname{arc} \operatorname{tg}(\sin x - 2\cos x) \end{aligned}$$

$$+ \frac{1}{10\sqrt{6}} \ln \left| \frac{\sqrt{6} + 2\sin x + \cos x}{\sqrt{6} - 2\sin x - \cos x} \right| + C.$$

2056. $\int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx.$

解 本题也是2053题的特例, 因为

$$\begin{aligned} & \int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx \\ &= \int \frac{\sin x - 2\cos x}{\sin^2 x + 4\sin x \cos x + \cos^2 x} dx, \end{aligned}$$

这里,

$$a_1 = 1, b_1 = 2, a = 1, b = 2, c = 1;$$

$$\lambda_1 = 3, \lambda_2 = -1; k_1 = -\frac{1}{2}, k_2 = \frac{1}{2};$$

$$A = \frac{1}{4}, B = -\frac{3}{4};$$

$$u_1 = 2(\cos x - \sin x), u_2 = 2(\cos x + \sin x).$$

代入得

$$\begin{aligned} & \int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx = \frac{1}{4} \int \frac{2d(\cos x - \sin x)}{-2(\cos x - \sin x)^2 + 3} \\ & \quad - \frac{3}{4} \int \frac{2d(\cos x + \sin x)}{2(\cos x + \sin x)^2 - 1} \\ &= \frac{3}{4\sqrt{2}} \ln \left| \frac{\sqrt{2}(\sin x + \cos x) + 1}{\sqrt{2}(\sin x + \cos x) - 1} \right| \\ & \quad - \frac{1}{4\sqrt{6}} \ln \left| \frac{\sqrt{3} + \sqrt{2}(\sin x - \cos x)}{\sqrt{3} - \sqrt{2}(\sin x - \cos x)} \right| + C. \end{aligned}$$

2057. 证明

$$\int \frac{dx}{(a \sin x + b \cos x)^n} = \frac{A \sin x + B \cos x}{(a \sin x + b \cos x)^{n-1}} + C \int \frac{dx}{(a \sin x + b \cos x)^{n-2}},$$

式中 A, B, C 为未定系数.

$$\text{证 } a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \alpha),$$

$$\text{式中 } \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}.$$

于是,

$$\begin{aligned} \int \frac{dx}{(a \sin x + b \cos x)^n} &= (a^2 + b^2)^{-\frac{n}{2}} \int \frac{dx}{\sin^n(x + \alpha)} \\ &= -(a^2 + b^2)^{-\frac{n}{2}} \int \frac{1}{\sin^{n-2}(x + \alpha)} d[\operatorname{ctg}(x + \alpha)] \\ &= -(a^2 + b^2)^{-\frac{n}{2}} \frac{\operatorname{ctg}(x + \alpha)}{\sin^{n-2}(x + \alpha)} \\ &\quad - \frac{n-2}{(a^2 + b^2)^{\frac{n}{2}}} \int \frac{\operatorname{ctg}(x + \alpha) \cos(x + \alpha)}{\sin^{n-1}(x + \alpha)} dx \\ &= \frac{\frac{b}{a^2 + b^2} \sin x - \frac{a}{a^2 + b^2} \cos x}{(a \sin x + b \cos x)^{n-1}} \\ &\quad - \frac{n-2}{(a^2 + b^2)^{\frac{n}{2}}} \int \frac{1 - \sin^2(x + \alpha)}{\sin^n(x + \alpha)} dx. \end{aligned}$$

设 $I_n = \int \frac{dx}{(a \sin x + b \cos x)^n}$, 则由上式可得

$$I_n = \frac{\frac{b}{a^2+b^2} \sin x - \frac{a}{a^2+b^2} \cos x}{(a \sin x + b \cos x)^{n-1}}$$

$$+ (2-n)I_n + \frac{n-2}{a^2+b^2} I_{n-2}.$$

于是,

$$I_n = \frac{\frac{b}{(n-1)(a^2+b^2)} \sin x - \frac{a}{(n-1)(a^2+b^2)} \cos x}{(a \sin x + b \cos x)^{n-1}}$$

$$+ \frac{n-2}{(n-1)(a^2+b^2)} I_{n-2}.$$

即

$$\int \frac{dx}{(a \sin x + b \cos x)^n} = \frac{A \sin x + B \cos x}{(a \sin x + b \cos x)^{n-1}}$$

$$+ C \int \frac{dx}{(a \sin x + b \cos x)^{n-2}},$$

$$\text{式中 } A = \frac{b}{(n-1)(a^2+b^2)}, \quad B = \frac{a}{(n-1)(a^2+b^2)},$$

$$C = \frac{n-2}{(n-1)(a^2+b^2)}.$$

2058. 求 $\int \frac{dx}{(\sin x + 2 \cos x)^3}$.

解 此为2057题之特例, 这里

$$a=1, b=2, n=3;$$

$$A = \frac{2}{10}, \quad B = -\frac{1}{10}, \quad C = \frac{1}{10}.$$

代入得

$$\begin{aligned} \int \frac{dx}{(\sin x + 2\cos x)^3} &= \frac{2\sin x - \cos x}{10(\sin x + 2\cos x)^2} \\ &+ \frac{1}{10} \int \frac{dx}{\sin x + 2\cos x} \\ &= \frac{2\sin x - \cos x}{10(\sin x + 2\cos x)^2} + \frac{1}{10\sqrt{5}} \int \frac{dx}{\sin(x + \alpha)} \\ &= \frac{2\sin x - \cos x}{10(\sin x + 2\cos x)^2} \\ &+ \frac{1}{10\sqrt{5}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\alpha}{2} \right) \right| + C. \end{aligned}$$

其中 $\cos \alpha = \frac{1}{\sqrt{5}}$, $\sin \alpha = \frac{2}{\sqrt{5}}$, $\alpha = \operatorname{arc} \operatorname{tg} 2$.

2059. 若 n 为大于 1 的自然数, 证明

$$\begin{aligned} \int \frac{dx}{(a + b\cos x)^n} &= \frac{A\sin x}{(a + b\cos x)^{n-1}} \\ &+ B \int \frac{dx}{(a + b\cos x)^{n-1}} + C \int \frac{dx}{(a + b\cos x)^{n-2}} \end{aligned}$$

($|a| \neq |b|$),

并求出系数 A, B 和 C .

证 设 $I_n = \int \frac{dx}{(a + b\cos x)^n}$, 先考虑 I_{n-1} .

$$I_{n-1} = \frac{1}{a} \int \frac{(a + b\cos x) - b\cos x}{(a + b\cos x)^{n-1}} dx$$

$$\begin{aligned}
&= \frac{1}{a} I_{n-2} - \frac{b}{a} \int \frac{d(\sin x)}{(a+b\cos x)^{n-1}} \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a+b\cos x)^{n-1}} \\
&\quad + \frac{(n-1)b^2}{a} \int \frac{\sin^2 x}{(a+b\cos x)^n} dx \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a+b\cos x)^{n-1}} \\
&\quad + \frac{n-1}{a} \int \frac{(b^2-a^2) + (a+b\cos x)(a-b\cos x)}{(a+b\cos x)^n} dx \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a+b\cos x)^{n-1}} + \frac{(n-1)(b^2-a^2)}{a} I_n \\
&\quad + \frac{n-1}{a} \int \frac{a-b\cos x}{(a+b\cos x)^{n-1}} dx \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a+b\cos x)^{n-1}} + \frac{(n-1)(b^2-a^2)}{a} I_n \\
&\quad - \frac{n-1}{a} \int \frac{(a+b\cos x) - 2a}{(a+b\cos x)^{n-1}} dx \\
&= \frac{1}{a} I_{n-2} - \frac{b\sin x}{a(a+b\cos x)^{n-1}} + \frac{(n-1)(b^2-a^2)}{a} I_n \\
&\quad - \frac{n-1}{a} I_{n-2} + 2(n-1) I_{n-1},
\end{aligned}$$

于是,

$$\frac{(n-1)(a^2-b^2)}{a} I_n = \frac{b\sin x}{a(a+b\cos x)^{n-1}}$$

$$+ (2n-3)I_{n-1} - \frac{n-2}{2}I_{n-2}.$$

最后得到

$$I_n = -\frac{b \sin x}{(n-1)(a^2-b^2)(a+b \cos x)^{n-1}} \\ + \frac{(2n-3)a}{(n-1)(a^2-b^2)}I_{n-1} - \frac{n-2}{(n-1)(a^2-b^2)}I_{n-2},$$

即

$$\int \frac{dx}{(a+b \cos x)^n} = \frac{A \sin x}{(a+b \cos x)^{n-1}} \\ + B \int \frac{dx}{(a+b \cos x)^{n-1}} + C \int \frac{dx}{(a+b \cos x)^{n-2}}.$$

式中 $A = -\frac{b}{(n-1)(a^2-b^2)}$, $B = \frac{(2n-3)a}{(n-1)(a^2-b^2)}$,

$$C = -\frac{n-2}{(n-1)(a^2-b^2)} \quad (|a| \neq |b|; n > 1 \text{ 且 } a \neq 0).$$

若 $a = 0$, 则 $b \neq 0$, 我们有

$$\int \frac{dx}{(a+b \cos x)^n} = \frac{1}{b^n} \int \frac{dx}{\cos^n x} \\ = \frac{1}{b^n} \left[\frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x} \right]^*.$$

*) 利用2012题(0)的结果.

求积分:

2060. $\int \frac{\sin x dx}{\cos x \sqrt{1+\sin^2 x}}.$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\sin x dx}{\cos x \sqrt{1 + \sin^2 x}} = \int \frac{-d(\cos x)}{\cos x \sqrt{2 - \cos^2 x}} \\
 & = - \int \frac{d(\cos x)}{\cos^2 x \sqrt{2 \sec^2 x - 1}} = \int \frac{d(\sec x)}{\sqrt{2 \sec^2 x - 1}} \\
 & = \frac{1}{\sqrt{2}} \ln \left| \sqrt{2} \sec x + \sqrt{2 \sec^2 x - 1} \right| + C \\
 & = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 + \sin^2 x}}{|\cos x|} + C.
 \end{aligned}$$

$$2061. \int \frac{\sin^2 x}{\cos^2 x \sqrt{\operatorname{tg} x}} dx.$$

$$\begin{aligned}
 \text{解} \quad & \int \frac{\sin^2 x}{\cos^2 x \sqrt{\operatorname{tg} x}} dx = \int \frac{\sin^2 x d(\operatorname{tg} x)}{\sqrt{\operatorname{tg} x}} \\
 & = 2 \int \sin^2 x d(\sqrt{\operatorname{tg} x}) = 2 \int (1 - \cos^2 x) d(\sqrt{\operatorname{tg} x}) \\
 & = 2\sqrt{\operatorname{tg} x} - 2 \int \frac{d(\sqrt{\operatorname{tg} x})}{1 + \operatorname{tg}^2 x} \\
 & = 2\sqrt{\operatorname{tg} x} - \frac{1}{2\sqrt{2}} \ln \frac{\operatorname{tg} x + \sqrt{2\operatorname{tg} x + 1}}{\operatorname{tg} x - \sqrt{2\operatorname{tg} x + 1}} \\
 & \quad + \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{\sqrt{2\operatorname{tg} x}}{\operatorname{tg} x - 1} + C \quad (\operatorname{tg} x > 0),
 \end{aligned}$$

*) 利用1884题的结果。

$$2062. \int \frac{\sin x dx}{\sqrt{2 + \sin 2x}}.$$

解 由于

$$2 + \sin 2x = 1 + (\sin x + \cos x)^2$$

$$= 3 - (\sin x - \cos x)^2.$$

于是,

$$\begin{aligned} \int \frac{\sin x dx}{\sqrt{2 + \sin 2x}} &= \int \frac{\cos x - (\cos x - \sin x)}{\sqrt{1 + (\sin x + \cos x)^2}} dx \\ &= \int \frac{\cos x dx}{\sqrt{3 - (\sin x - \cos x)^2}} dx \\ &\quad - \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}) \\ &= - \int \frac{\sin x dx}{\sqrt{2 + \sin 2x}} + \int \frac{d(\sin x - \cos x)}{\sqrt{3 - (\sin x - \cos x)^2}} \\ &\quad - \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}), \end{aligned}$$

因而,

$$\begin{aligned} \int \frac{\sin x dx}{\sqrt{2 + \sin 2x}} &= \frac{1}{2} \int \frac{d(\sin x - \cos x)}{\sqrt{3 - (\sin x - \cos x)^2}} \\ &\quad - \frac{1}{2} \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}) \\ &= \frac{1}{2} \arcsin\left(\frac{\sin x - \cos x}{\sqrt{3}}\right) \\ &\quad - \frac{1}{2} \ln(\sin x + \cos x + \sqrt{2 + \sin 2x}) + C. \end{aligned}$$

2063. $\int \frac{dx}{(1 + \varepsilon \cos x)^2} \quad (0 < \varepsilon < 1).$

解 此为2059题之特例, 这里

$$a = 1, \quad b = \varepsilon, \quad n = 2,$$

$$A = -\frac{\varepsilon}{1 - \varepsilon^2}, \quad B = \frac{1}{1 - \varepsilon^2}, \quad C = 0.$$

代入得

$$\begin{aligned} & \int \frac{dx}{(1+e\cos x)^2} = -\frac{e\sin x}{(1-e^2)(1+e\cos x)} \\ & + \frac{1}{1-e^2} \int \frac{1}{1+e\cos x} \\ & = -\frac{e\sin x}{(1-e^2)(1+e\cos x)} \\ & + \frac{2}{(1-e^2)^{\frac{3}{2}}} \operatorname{arc\,tg} \left(\sqrt{\frac{1-e}{1+e}} \operatorname{tg} \frac{x}{2} \right)^* + C. \end{aligned}$$

*) 利用2028题(a)的结果.

$$2064^+. \int \frac{\cos^{n-1} \frac{x+a}{2}}{\sin^{n+1} \frac{x-a}{2}} dx$$

解 设 $t = \frac{\cos \frac{x+a}{2}}{\sin \frac{x-a}{2}}$, 则

$$dt = \frac{-\frac{1}{2} \cos a}{\sin^2 \frac{x-a}{2}} dx, \quad \frac{dx}{\sin^2 \frac{x-a}{2}} = -\frac{2}{\cos a} dt.$$

于是,

$$\int \frac{\cos^{n-1} \frac{x+a}{2}}{\sin^{n+1} \frac{x-a}{2}} dx = -\frac{2}{\cos a} \int t^{n-1} dt$$

$$= -\frac{2}{n \cos a} t^n + C$$

$$= -\frac{2}{n \cos a} \left(\frac{\cos \frac{x+a}{2}}{\sin \frac{x-a}{2}} \right)^n + C \quad (\cos a \neq 0).$$

2065. 推出积分

$$I_n = \int \left(\frac{\sin \frac{x-a}{2}}{\sin \frac{x+a}{2}} \right)^n dx$$

的递推公式 (n 为自然数)。

证 方法一:

$$\text{设 } t = \frac{\sin \frac{x-a}{2}}{\sin \frac{x+a}{2}}, \text{ 则}$$

$$x = 2 \arctg \left(\frac{1+t}{1-t} \cdot \operatorname{tg} \frac{a}{2} \right),$$

$$dx = \frac{4 \operatorname{tg} \frac{a}{2}}{t^2 \sec^2 \frac{a}{2} + 2t \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right) + \sec^2 \frac{a}{2}} dt.$$

由于

$$\frac{4t^n \operatorname{tg} \frac{a}{2}}{t^2 \sec^2 \frac{a}{2} + 2t \left(\operatorname{tg}^2 \frac{a}{2} - 1 \right) + \sec^2 \frac{a}{2}}$$

$$\begin{aligned}
&= \frac{4\operatorname{tg}\frac{a}{2}}{\sec^2\frac{a}{2}} t^{n-2} + \frac{-4\operatorname{tg}\frac{a}{2}}{t^2\sec^2\frac{a}{2} + 2t\left(\operatorname{tg}^2\frac{a}{2} - 1\right) + \sec^2\frac{a}{2}} \\
&\quad \cdot \frac{2\left(\operatorname{tg}^2\frac{a}{2} - 1\right)}{\sec^2\frac{a}{2}} t^{n-1} \\
&\quad + \frac{-4\operatorname{tg}\frac{a}{2}}{t^2\sec^2\frac{a}{2} + 2t\left(\operatorname{tg}^2\frac{a}{2} - 1\right) + \sec^2\frac{a}{2}} \\
&\quad \cdot t^{n-2} \quad (n > 2),
\end{aligned}$$

两端对 t 积分, 即得递推公式

$$I_n = \frac{2\sin a}{n-1} t^{n-1} + 2I_{n-1} \cos a - I_{n-2}.$$

方法二:

设 $y = \frac{x+a}{2}$, 则 $\frac{x-a}{2} = y-a$, 从而

$$\begin{aligned}
I_n &= 2 \int \left[\frac{\sin(y-a)}{\sin y} \right]^n dy \\
&= 2 \int \frac{\sin(y-a)}{\sin y} \left[\frac{\sin(y-a)}{\sin y} \right]^{n-1} dy \\
&= 2 \int \frac{\sin y \cos a - \cos y \sin a}{\sin y} \left[\frac{\sin(y-a)}{\sin y} \right]^{n-1} dy \\
&= \cos a I_{n-1} - 2 \sin a \int \frac{\cos y}{\sin y} \left[\frac{\sin(y-a)}{\sin y} \right]^{n-1} dy.
\end{aligned}$$

再设

$$\frac{\sin(y-a)}{\sin y} = t = \frac{\sin\left(\frac{x-a}{2}\right)}{\sin\frac{x+a}{2}}, \quad J_n = 2 \int \frac{\cos y}{\sin y} t^n dy,$$

则

$$I_n = \cos a I_{n-1} - \sin a J_{n-1}, \quad J_{n-1} = \frac{\cos a I_{n-1} - I_n}{\sin a}. \quad (1)$$

又

$$\begin{aligned} J_n &= 2 \int \frac{\cos y}{\sin y} \left[\frac{\sin(y-a)}{\sin y} \right]^n dy \\ &= -\frac{2}{n} \int \sin^n(y-a) d\left(\frac{1}{\sin^n y}\right) \\ &= -\frac{2}{n} \left[\frac{\sin(y-a)}{\sin y} \right]^n \\ &\quad + 2 \int \frac{\sin^{n-1}(y-a)}{\sin^n y} \cos(y-a) dy \\ &= -\frac{2}{n} t^n + 2 \int t^{n-1} \frac{\cos y \cos a + \sin y \sin a}{\sin y} dy \\ &= -\frac{2}{n} t^n + \cos a J_{n-1} + \sin a I_{n-1}. \quad (2) \end{aligned}$$

由(1)式和(2)式解得

$$\begin{aligned} I_n &= I_{n-1} \cos a - \sin a J_{n-1} \\ &= I_{n-1} \cos a - \sin a \left(-\frac{2}{n-1} t^{n-1} + \cos a J_{n-2} + I_{n-2} \sin a \right) \\ &= I_{n-1} \cos a + \frac{2 \sin a}{n-1} t^{n-1} \end{aligned}$$

$$\begin{aligned}
& -\sin a \cos a \left(\frac{I_{n-2} \cos a - I_{n-1}}{\sin a} \right) - \sin^2 a I_{n-2} \\
& = 2I_{n-1} \cos a - I_{n-2} + \frac{2 \sin a}{n-1} I^{n-1}.
\end{aligned}$$

§5. 各种超越函数的积分法

2066. 证明若 $P(x)$ 为 n 次多项式, 则

$$\begin{aligned}
\int P(x) e^{ax} dx &= e^{ax} \left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} \right. \\
&\quad \left. + \dots + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}} \right] + C.
\end{aligned}$$

证
$$\begin{aligned}
\int P(x) e^{ax} dx &= \frac{1}{a} \int P(x) d(e^{ax}) \\
&= \frac{1}{a} P(x) e^{ax} - \frac{1}{a} \int e^{ax} P'(x) dx \\
&= \frac{1}{a} P(x) e^{ax} - \frac{1}{a^2} \int P'(x) d(e^{ax}) \\
&= \frac{1}{a} P(x) e^{ax} - \frac{1}{a^2} P'(x) e^{ax} + \frac{1}{a^2} \int e^{ax} P''(x) dx \\
&= \dots \\
&= e^{ax} \left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} + \dots + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}} \right] + C.
\end{aligned}$$

因为 $P(x)$ 为 n 次多项式, 所以 $P^{(n+1)}(x) \equiv 0$, 从而上述等式括号中的导数到 $P^{(n)}(x)$ 为止.

2067. 证明若 $P(x)$ 为 n 次多项式, 则

$$\int P(x) \cos ax dx = \frac{\sin ax}{a} \left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots \right] + \frac{\cos ax}{a^2} \left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots \right] + C$$

及

$$\int P(x) \sin ax dx = -\frac{\cos ax}{a} \left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots \right] + \frac{\sin ax}{a^2} \left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots \right] + C.$$

证
$$\begin{aligned} \int P(x) \cos ax dx &= \frac{1}{a} \int P(x) d(\sin ax) \\ &= \frac{1}{a} P(x) \sin ax - \frac{1}{a} \int P'(x) \sin ax dx \\ &= \frac{1}{a} P(x) \sin ax + \frac{1}{a^2} \int P'(x) d(\cos ax) \\ &= \frac{1}{a} P(x) \sin ax + \frac{1}{a^2} P'(x) \cos ax \\ &\quad - \frac{1}{a^2} \int P''(x) \cos ax dx \\ &= \frac{1}{a} P(x) \sin ax + \frac{1}{a^2} P'(x) \cos ax - \frac{1}{a^3} P''(x) \sin ax \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{a^4}P'''(x)\cos ax + \frac{1}{a^4}\int P^{(4)}(x)\cos ax dx \\
& = \dots \\
& = \frac{\sin ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots\right] \\
& + \frac{\cos ax}{a^2}\left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{(5)}(x)}{a^4} - \dots\right] + C.
\end{aligned}$$

$$\begin{aligned}
\int P(x)\sin ax dx &= -\frac{1}{a}\int P(x)d(\cos ax) \\
&= -\frac{1}{a}P(x)\cos ax + \frac{1}{a}\int P'(x)\cos ax dx \\
&= -\frac{1}{a}P(x)\cos ax + \frac{1}{a^2}\int P'(x)d(\sin ax) \\
&= -\frac{1}{a}P(x)\cos ax + \frac{1}{a^2}P'(x)\sin ax \\
&\quad - \frac{1}{a^2}\int P''(x)\sin ax dx \\
&= -\frac{1}{a}P(x)\cos ax + \frac{1}{a^2}P'(x)\sin ax + \frac{1}{a^3}P''(x)\cos ax \\
&\quad - \frac{1}{a^4}P'''(x)\sin ax + \frac{1}{a^4}\int P^{(4)}(x)\sin ax dx \\
&= \dots \\
&= -\frac{\cos ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{(4)}(x)}{a^4} - \dots\right]
\end{aligned}$$

$$+ \frac{\sin ax}{a^2} \left[P^{(4)}(x) - \frac{P^{(5)}(x)}{a^2} + \frac{P^{(6)}(x)}{a^4} - \dots \right] + C.$$

上述导数项是有限的，其次数 $\leq n$ 。

求积分：

2068. $\int x^3 e^{3x} dx.$

解 $\int x^3 e^{3x} dx = e^{3x} \left(\frac{x^3}{3} - \frac{3x^2}{9} + \frac{6x}{27} - \frac{6}{81} \right)^{*}$ + C

$$= e^{3x} \left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{2}{27} \right) + C.$$

*) 利用2066题的结果。

2069. $\int (x^2 - 2x + 2) e^{-x} dx.$

解 $\int (x^2 - 2x + 2) e^{-x} dx = e^{-x} \left(\frac{x^2 - 2x + 2}{-1} \right.$

$$\left. - \frac{2x - 2}{1} + \frac{2}{-1} \right)^{*}$$
 + C

$$= -e^{-x}(x^2 + 2) + C.$$

*) 利用2066题的结果。

2070. $\int x^5 \sin 5x dx.$

解 $\int x^5 \sin 5x dx = -\frac{\cos 5x}{5} \left(x^5 - \frac{20x^3}{25} + \frac{120x}{625} \right)$

$$+ \frac{\sin 5x}{25} \left(5x^4 - \frac{60x^2}{25} + \frac{120}{625} \right)^{*}$$
 + C

$$= -\frac{\cos 5x}{5} \left(x^5 - \frac{4x^3}{5} + \frac{24x}{125} \right)$$

$$+ \frac{\sin 5x}{25} \left(5x^4 - \frac{12x^2}{5} + \frac{24}{125} \right) + C.$$

*) 利用2067题的结果。

$$2071. \int (1+x^2)^2 \cos x dx.$$

$$\begin{aligned} \text{解} \quad \int (1+x^2)^2 \cos x dx &= \int (1+2x^2+x^4) \cos x dx \\ &= \sin x \left[(1+2x^2+x^4) - (4+12x^2) + 24 \right] \\ &\quad + \cos x \left[(4x+4x^3) - 24x \right]^* + C \\ &= (21-10x^2+x^4) \sin x - (20x-4x^3) \cos x + C. \end{aligned}$$

*) 利用2067题的结果。

$$2072. \int x^7 e^{-x^2} dx.$$

$$\begin{aligned} \text{解} \quad \int x^7 e^{-x^2} dx &= \frac{1}{2} \int (x^2)^3 e^{-x^2} d(x^2) \\ &= \frac{1}{2} e^{-x^2} \left(\frac{x^6}{-1} - \frac{3x^4}{1} + \frac{6x^2}{-1} - \frac{6}{1} \right)^* + C \\ &= -\frac{1}{2} e^{-x^2} (x^6 + 3x^4 + 6x^2 + 6) + C. \end{aligned}$$

*) 利用2066题的结果。

$$2073. \int x^2 e^{\sqrt{x}} dx.$$

$$\begin{aligned} \text{解} \quad \int x^2 e^{\sqrt{x}} dx &= 2 \int (\sqrt{x})^4 e^{\sqrt{x}} d(\sqrt{x}) \\ &= 2e^{\sqrt{x}} \left(x^{\frac{5}{2}} - 5x^2 + 20x^{\frac{3}{2}} - 60x + 120x^{\frac{1}{2}} - 120 \right)^* + C. \end{aligned}$$

*) 利用2066题的结果。

$$2074. \int e^{ax} \cos^2 bx dx.$$

$$\begin{aligned} \text{解} \quad \int e^{ax} \cos^2 bx dx &= \frac{1}{2} \int e^{ax} (1 + \cos 2bx) dx \\ &= \frac{1}{2a} e^{ax} + \frac{1}{2} e^{ax} \cdot \frac{a \cos 2bx + 2b \sin 2bx}{a^2 + 4b^2} + C. \end{aligned}$$

*) 利用1828题的结果。

$$2075. \quad \int e^{ax} \sin^3 bx dx.$$

$$\begin{aligned} \text{解} \quad \int e^{ax} \sin^3 bx dx &= \int e^{ax} \sin bx \cdot \frac{1 - \cos 2bx}{2} dx \\ &= \int e^{ax} \left(\frac{3}{4} \sin bx - \frac{1}{4} \sin 3bx \right) dx \\ &= \frac{3}{4} e^{ax} \cdot \frac{a \sin bx - b \cos bx}{a^2 + b^2} \\ &\quad - \frac{1}{4} e^{ax} \cdot \frac{a \sin 3bx - 3b \cos 3bx}{a^2 + 9b^2} + C. \end{aligned}$$

*) 利用1829题的结果。

$$2076. \quad \int x e^x \sin x dx.$$

$$\begin{aligned} \text{解} \quad \int x e^x \sin x dx &= \int x \sin x d(e^x) \\ &= x e^x \sin x - \int e^x (\sin x + x \cos x) dx \\ &= x e^x \sin x - \int (\sin x + x \cos x) d(e^x) \\ &= e^x (x \sin x - \sin x - x \cos x) \\ &\quad + \int e^x (2 \cos x - x \sin x) dx \end{aligned}$$

$$= e^x(x \sin x - \sin x - x \cos x) \\ + 2 \int e^x \cos x dx - \int x e^x \sin x dx.$$

于是,

$$\int x e^x \sin x dx = \frac{e^x}{2}(x \sin x - \sin x \\ - x \cos x) + \int e^x \cos x dx \\ = \frac{e^x}{2}(x \sin x - \sin x - x \cos x) + \frac{e^x}{2}(\sin x + \cos x) + C^*) \\ = \frac{e^x}{2}[x(\sin x - \cos x) + \cos x] + C.$$

*) 利用1828题的结果.

2077. $\int x^2 e^x \cos x dx.$

解 $\int x^2 e^x \cos x dx = \int x^2 \cos x d(e^x)$

$$= x^2 e^x \cos x - \int e^x (2x \cos x - x^2 \sin x) dx$$

$$= x^2 e^x \cos x - \int (2x \cos x - x^2 \sin x) d(e^x)$$

$$= x^2 e^x \cos x - e^x (2x \cos x - x^2 \sin x) \\ + \int e^x (2 \cos x - 4x \sin x - x^2 \cos x) dx$$

$$= e^x [x^2 (\sin x + \cos x) - 2x \cos x] + 2 \int e^x \cos x dx \\ - 4 \int x e^x \sin x dx - \int x^2 e^x \cos x dx,$$

于是,

$$\int x^2 e^x \cos x dx = \frac{e^x}{2} [x^2 (\sin x + \cos x) - 2x \cos x] \\ + \int e^x \cos x dx - 2 \int x e^x \sin x dx \\ = \frac{e^x}{2} [x^2 (\sin x + \cos x) - 2x \cos x] + \frac{e^x}{2} (\sin x + \cos x)^* \\ - 2 \cdot \frac{e^x}{2} [x (\sin x - \cos x) + \cos x]^{**} + C \\ = \frac{e^x}{2} [x^2 (\sin x + \cos x) - 2x \sin x \\ + (\sin x - \cos x)] + C.$$

*) 利用1828题的结果.

***) 利用2076题的结果.

2078. $\int x e^x \sin^2 x dx.$

解 $\int x e^x \sin^2 x dx = \frac{1}{2} \int x e^x (1 - \cos 2x) dx \\ = \frac{1}{2} \int x e^x dx - \frac{1}{2} \int x e^x \cos 2x dx \\ = \frac{1}{2} e^x (x - 1) - \frac{1}{2} \int x \cos 2x d(e^x)$

$$\begin{aligned}
&= \frac{1}{2}e^x(x-1) - \frac{1}{2}xe^x\cos 2x \\
&\quad + \frac{1}{2}\int e^x(\cos 2x - 2x\sin 2x)dx \\
&= \frac{1}{2}e^x(x-1) - \frac{1}{2}xe^x\cos 2x + \frac{e^x}{2} \cdot \frac{\cos 2x + 2\sin 2x^{**})}{5} \\
&\quad - \int xe^x\sin 2x dx,
\end{aligned}$$

而

$$\begin{aligned}
\int xe^x\sin 2x dx &= \int x\sin 2x d(e^x) \\
&= xe^x\sin 2x - \int e^x(\sin 2x + 2x\cos 2x)dx \\
&= xe^x\sin 2x - \frac{e^x}{5}(\sin 2x - 2\cos 2x)^{**}) \\
&\quad - 2\int xe^x(1 - 2\sin^2 x)dx \\
&= xe^x\sin 2x - \frac{e^x}{5}(\sin 2x - 2\cos 2x) \\
&\quad - 2(x-1)e^x + 4\int xe^x\sin^2 x dx.
\end{aligned}$$

代入得

$$\begin{aligned}
\int xe^x\sin^2 x dx &= e^x \left[\frac{x-1}{2} - \frac{x}{10}(2\sin 2x + \cos 2x) \right. \\
&\quad \left. + \frac{1}{50}(4\sin 2x - 3\cos 2x) \right] + C.
\end{aligned}$$

*) 利用1828题的结果.

***) 利用1829题的结果.

$$2079. \int (x - \sin x)^3 dx.$$

$$\text{解} \quad \int (x - \sin x)^3 dx$$

$$= \int (x^3 - 3x^2 \sin x + 3x \sin^2 x - \sin^3 x) dx$$

$$= \frac{x^4}{4} + 3 \int x^2 d(\cos x) + \frac{3}{2} \int x(1 - \cos 2x) dx$$

$$+ \int (1 - \cos^2 x) d(\cos x)$$

$$= \frac{x^4}{4} + 3x^2 \cos x - 6 \int x \cos x dx + \frac{3}{4} x^2$$

$$- \frac{3}{4} \int x d(\sin 2x) + \cos x - \frac{1}{3} \cos^3 x$$

$$= \frac{x^4}{4} + \frac{3x^2}{4} + 3x^2 \cos x - 6 \int x d(\sin x) - \frac{3}{4} x \sin 2x$$

$$+ \frac{3}{4} \int \sin 2x dx + \cos x - \frac{1}{3} \cos^3 x$$

$$= \frac{x^4}{4} + \frac{3x^2}{4} + 3x^2 \cos x - 6x \sin x - 6 \cos x$$

$$- \frac{3}{4} x \sin 2x + \cos x - \frac{3}{8} \cos 2x - \frac{1}{3} \cos^3 x + C$$

$$= \frac{x^4}{4} + \frac{3x^2}{4} + 3x^2 \cos x - x \left(6 \sin x + \frac{3}{4} \sin 2x \right)$$

$$-\left(5\cos x + \frac{3}{8}\cos 2x\right) - \frac{1}{3}\cos^3 x + C.$$

2080. $\int \cos^2 \sqrt{x} dx.$

解 设 $\sqrt{x} = t$, 则 $x = t^2$, $dx = 2t dt$. 于是,

$$\begin{aligned} \int \cos^2 \sqrt{x} dx &= 2 \int t \cos^2 t dt = \int t(1 + \cos 2t) dt \\ &= \frac{t^2}{2} + \frac{1}{2} \int t d(\sin 2t) \\ &= \frac{t^2}{2} + \frac{1}{2} t \sin 2t - \frac{1}{2} \int \sin 2t dt \\ &= \frac{t^2}{2} + \frac{1}{2} t \sin 2t + \frac{1}{4} \cos 2t + C \\ &= \frac{x}{2} + \frac{1}{2} \sqrt{x} \sin(2\sqrt{x}) + \frac{1}{4} \cos(2\sqrt{x}) + C. \end{aligned}$$

2081. 证明若 R 为有理函数及 a_1, a_2, \dots, a_n 为可公度的数, 则积分

$$\int R(e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}) dx$$

是初等函数.

证 按题意 a_1, a_2, \dots, a_n 为可公度的数, 于是存在一个实数 α , 使得

$$a_1 = k_1 \alpha, a_2 = k_2 \alpha, \dots, a_n = k_n \alpha \quad (\alpha \neq 0),$$

其中 k_1, k_2, \dots, k_n 为整数.

$$\text{设 } e^{\alpha x} = t, \text{ 则 } x = \frac{1}{\alpha} \ln t, \quad dx = \frac{1}{\alpha t} dt.$$

于是,

$$\begin{aligned} & \int R(e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}) dx \\ &= \frac{1}{a} \int R(t^{k_1}, t^{k_2}, \dots, t^{k_n}) \frac{dt}{t} = \int R^*(t) dt, \end{aligned}$$

其中 $R^*(t)$ 是 t 的有理函数. 因此, 积分

$$\int R(e^{a_1 x}, e^{a_2 x}, \dots, e^{a_n x}) dx$$

为初等函数.

求下列积分:

2082. $\int \frac{dx}{(1+e^x)^2}$.

解
$$\begin{aligned} \int \frac{dx}{(1+e^x)^2} &= \int \frac{(1+e^x) - e^x}{(1+e^x)^2} dx \\ &= \int \frac{dx}{1+e^x} - \int \frac{e^x dx}{(1+e^x)^2} \\ &= \int \left(1 - \frac{e^x}{1+e^x}\right) dx - \int \frac{d(1+e^x)}{(1+e^x)^2} \\ &= x - \ln(1+e^x) + \frac{1}{1+e^x} + C. \end{aligned}$$

2083. $\int \frac{e^{2x} dx}{1+e^x}$.

解
$$\begin{aligned} \int \frac{e^{2x} dx}{1+e^x} &= \int \frac{(e^{2x}-1)+1}{1+e^x} dx \\ &= \int (e^x-1) dx + \int \frac{1}{1+e^x} dx \end{aligned}$$

$$= e^x - x + \int \left(1 - \frac{e^x}{1+e^x}\right) dx = e^x - \ln(1+e^x) + C.$$

2084. $\int \frac{dx}{e^{2x} + e^x - 2}$.

解 $\int \frac{dx}{e^{2x} + e^x - 2} = \int \frac{dx}{(e^x + 2)(e^x - 1)}$

$$= \frac{1}{3} \int \frac{1}{e^x - 1} dx - \frac{1}{3} \int \frac{1}{e^x + 2} dx$$

$$= -\frac{1}{3} \int \left(1 - \frac{e^x}{e^x - 1}\right) dx - \frac{1}{6} \int \left(1 - \frac{e^x}{e^x + 2}\right) dx$$

$$= -\frac{x}{3} + \frac{1}{3} \ln|e^x - 1| - \frac{x}{6} + \frac{1}{6} \ln(e^x + 2) + C$$

$$= -\frac{x}{2} + \frac{1}{3} \ln|e^x - 1| + \frac{1}{6} \ln(e^x + 2) + C.$$

2085. $\int \frac{dx}{1 + e^{\frac{x}{2}} + e^{\frac{x}{3}} + e^{\frac{x}{6}}}$.

解 设 $e^{\frac{x}{6}} = t$, 则 $x = 6 \ln t$, $dx = \frac{6}{t} dt$.

代入得

$$\int \frac{dx}{1 + e^{\frac{x}{2}} + e^{\frac{x}{3}} + e^{\frac{x}{6}}} = 6 \int \frac{dt}{t(1 + t^3 + t^2 + t)}$$

$$= 6 \int \frac{dt}{t(t+1)(t^2+1)}$$

$$\begin{aligned}
&= 6 \int \left[\frac{1}{t} - \frac{1}{2(t+1)} - \frac{t+1}{2(t^2+1)} \right] dt \\
&= 6 \ln t - 3 \ln(t+1) - \frac{3}{2} \ln(1+t^2) - 3 \operatorname{arctg} t + C \\
&= x - 3 \ln \left[(1+e^{\frac{x}{6}}) \sqrt{1+e^{\frac{x}{3}}} \right] - 3 \operatorname{arctg}(e^{\frac{x}{6}}) + C.
\end{aligned}$$

2086. $\int \frac{1+e^{\frac{x}{2}}}{(1+e^{\frac{x}{4}})^2} dx.$

解 设 $e^{\frac{x}{4}} = t$, 则 $x = 4 \ln t$, $dx = \frac{4}{t} dt$.

代入得

$$\begin{aligned}
\int \frac{1+e^{\frac{x}{2}}}{(1+e^{\frac{x}{4}})^2} dx &= 4 \int \frac{1+t^2}{t(1+t)^2} dt \\
&= 4 \int \left[\frac{1}{t} - \frac{2}{(1+t)^2} \right] dt \\
&= 4 \ln t + \frac{8}{1+t} + C = x + \frac{8}{1+e^{\frac{x}{4}}} + C.
\end{aligned}$$

2087. $\int \frac{dx}{\sqrt{e^x-1}}.$

$$\begin{aligned}
\text{解 } \int \frac{dx}{\sqrt{e^x-1}} &= \int \frac{dx}{e^{\frac{x}{2}} \sqrt{1-(e^{-\frac{x}{2}})^2}} \\
&= -2 \int \frac{d(e^{-\frac{x}{2}})}{\sqrt{1-(e^{-\frac{x}{2}})^2}}
\end{aligned}$$

$$= -2 \arcsin \left(e^{-\frac{x}{2}} \right) + C.$$

2088. $\int \sqrt{\frac{e^x-1}{e^x+1}} dx.$

$$\begin{aligned} \text{解} \quad \int \sqrt{\frac{e^x-1}{e^x+1}} dx &= \int \frac{e^x-1}{\sqrt{e^{2x}-1}} dx \\ &= \int \frac{e^x dx}{\sqrt{e^{2x}-1}} - \int \frac{dx}{\sqrt{e^{2x}-1}} \\ &= \int \frac{d(e^x)}{\sqrt{(e^x)^2-1}} + \int \frac{d(e^{-x})}{\sqrt{1-(e^{-x})^2}} \\ &= \ln(e^x + \sqrt{e^{2x}-1}) + \arcsin(e^{-x}) + C. \end{aligned}$$

2089. $\int \sqrt{e^{2x}+4e^x-1} dx.$

$$\begin{aligned} \text{解} \quad \int \sqrt{e^{2x}+4e^x-1} dx &= \int \frac{e^{2x}+4e^x-1}{\sqrt{e^{2x}+4e^x-1}} dx \\ &= \int \frac{2e^{2x}+4e^x}{2\sqrt{e^{2x}+4e^x-1}} dx \\ &\quad + 2 \int \frac{e^x dx}{\sqrt{e^{2x}+4e^x-1}} - \int \frac{dx}{\sqrt{e^{2x}+4e^x-1}} \\ &= \int \frac{d(e^{2x}+4e^x-1)}{2\sqrt{e^{2x}+4e^x-1}} + 2 \int \frac{d(e^x+2)}{\sqrt{(e^x+2)^2-5}} \\ &\quad + \int \frac{d(e^{-x}-2)}{\sqrt{5-(e^{-x}-2)^2}} \\ &= \sqrt{e^{2x}+4e^x-1} + 2 \ln(e^x+2+\sqrt{e^{2x}+4e^x-1}) \end{aligned}$$

$$-\arcsin \frac{2e^x - 1}{\sqrt{5}e^x} + C.$$

2090. $\int \frac{dx}{\sqrt{1+e^x} + \sqrt{1-e^x}}.$

解 $\int \frac{dx}{\sqrt{1+e^x} + \sqrt{1-e^x}}$

$$= \frac{1}{2} \int e^{-x} (\sqrt{1+e^x} - \sqrt{1-e^x}) dx$$

$$= -\frac{1}{2} \int (\sqrt{1+e^x} - \sqrt{1-e^x}) d(e^{-x})$$

$$= -\frac{e^{-x}}{2} (\sqrt{1+e^x} - \sqrt{1-e^x})$$

$$+ \frac{1}{4} \int \left(\frac{1}{\sqrt{1+e^x}} + \frac{1}{\sqrt{1-e^x}} \right) dx$$

$$= -\frac{1}{2} e^{-x} (\sqrt{1+e^x} - \sqrt{1-e^x}) + \frac{1}{4} I_1 + \frac{1}{4} I_2.$$

对于 $I_1 = \int \frac{dx}{\sqrt{1+e^x}}$, 设 $\sqrt{1+e^x} = t$, 则

$$x = \ln(t^2 - 1), \quad dx = \frac{2t dt}{t^2 - 1}.$$

于是,

$$I_1 = \int \frac{dx}{\sqrt{1+e^x}} = 2 \int \frac{dt}{t^2 - 1}$$

$$= \ln \frac{t-1}{t+1} + C = \ln \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} + C_1.$$

对于 $I_2 = \int \frac{dx}{\sqrt{1-e^x}}$, 设 $\sqrt{1-e^x} = t$, 则

$$x = \ln(1-t^2), \quad dx = -\frac{2t dt}{1-t^2}.$$

于是,

$$\begin{aligned} I_2 &= \int \frac{dx}{\sqrt{1-e^x}} = -2 \int \frac{dt}{1-t^2} = -\ln \frac{1+t}{1-t} + C_2 \\ &= -\ln \frac{1+\sqrt{1-e^x}}{1-\sqrt{1-e^x}} + C_2. \end{aligned}$$

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt{1+e^x} + \sqrt{1-e^x}} &= -\frac{e^{-x}}{2} (\sqrt{1+e^x} - \sqrt{1-e^x}) \\ &+ \frac{1}{4} \ln \frac{(\sqrt{1+e^x} - 1)(1 - \sqrt{1-e^x})}{(\sqrt{1+e^x} + 1)(1 + \sqrt{1-e^x})} + C. \end{aligned}$$

2091. 证明积分

$$\int R(x)e^{ax} dx,$$

(式中 R 为有理函数, 其分母仅有实根) 可用初等函数和超越函数

$$\int \frac{e^{ax}}{x} dx = li(e^{ax}) + C, \quad \text{式中 } li x = \int \frac{dx}{\ln x}$$

来表示.

证 因为 R 的分母仅有实根, 所以仅包含形如 $(x-a_i)^{k_i}$ 的因子 ($i=1, 2, \dots, l$). 分解 $R(x)$ 为部分分式得

$$R(x) = P(x) + \sum_{i=1}^l \sum_{j=1}^{k_i} \frac{A_{ij}}{(x-a_i)^j},$$

其中 $P(x)$ 为 x 的多项式, A_{ij} 是常系数.

从而积分

$$\begin{aligned} \int R(x)e^{ax} dx &= \int P(x)e^{ax} dx \\ &+ \sum_{i=1}^l \sum_{j=1}^{k_i} A_{ij} \int \frac{e^{ax}}{(x-a_i)^j} dx. \end{aligned}$$

上式右端第一个积分显然是初等函数, 而积分

$$\int \frac{e^{ax}}{(x-a_i)^j} dx$$

可用初等函数和超越函数来表示. 事实上, 设 $x-a_i = t$, 则

$$\begin{aligned} \int \frac{e^{ax}}{(x-a_i)^j} dx &= \int \frac{e^{a(a_i+t)}}{t^j} dt \\ &= \frac{e^{aa_i}}{1-j} \int e^{at} d\left(-\frac{1}{t^{j-1}}\right) \\ &= \frac{e^{aa_i}}{1-j} e^{at} \cdot \frac{1}{t^{j-1}} - \frac{ae^{aa_i}}{1-j} \int \frac{e^{at}}{t^{j-1}} dt. \end{aligned}$$

这样, 被积函数中分母的次数便降低一次, 再继续运用分部积分法 ($j-2$) 次, 即可得

$$\int \frac{e^{ax}}{(x-a_i)^j} dx = g_{ij}(x) + B_{ij} \operatorname{li}(e^{a(x-a_i)}),$$

其中 $g_{ij}(x)$ 为 x 的初等函数, B_{ij} 为常数.

因而积分

$$\int R(x) e^{ax} dx = \int P(x) e^{ax} dx + \sum_{i=1}^l \sum_{j=1}^{k_i} A_{ij} g_{ij}(x) + \sum_{i=1}^l \sum_{j=1}^{k_i} A_{ij} B_{ij} \operatorname{li}(e^{a(x-a_i)})$$

是初等函数与超越函数之和。

2092. 在甚么情形下, 积分

$$\int P\left(\frac{1}{x}\right) e^x dx$$

(式中 $P\left(\frac{1}{x}\right) = a_0 + \frac{a_1}{x} + \dots + \frac{a_n}{x^n}$ 及 a_0, a_1, \dots, a_n 为常数) 为初等函数?

$$\begin{aligned} \text{解} \quad \int \frac{a_k}{x^k} e^x dx &= -\frac{a_k}{k-1} \cdot \frac{e^x}{x^{k-1}} + \frac{a_k}{k-1} \int \frac{e^x}{x^{k-1}} dx \\ &= \dots = -\frac{a_k}{k-1} \cdot \frac{e^x}{x^{k-1}} - \frac{a_k}{(k-1)(k-2)} \cdot \frac{e^x}{x^{k-2}} - \dots \\ &\quad - \frac{a_k}{(k-1)!} \cdot \frac{e^x}{x} + \frac{a_k}{(k-1)!} \int \frac{e^x}{x} dx, \end{aligned}$$

于是,

$$\begin{aligned} \int P\left(\frac{1}{x}\right) e^x dx &= \int \left(\sum_{k=0}^n \frac{a_k}{x^k} \right) e^x dx = \sum_{k=0}^n \int \frac{a_k}{x^k} dx \\ &= - \sum_{k=2}^n \sum_{j=1}^{k-1} \frac{a_k}{(k-1)(k-2)\dots(k-j)} \cdot \frac{e^x}{x^{k-j}} \end{aligned}$$

$$+ \sum_{k=1}^n \frac{a_k}{(k-1)!} \int \frac{e^x}{x} dx + a_0 e^x.$$

因而, 若

$$\sum_{k=1}^n \frac{a_k}{(k-1)!} = 0,$$

即

$$a_1 + \frac{a_2}{1!} + \frac{a_3}{2!} + \cdots + \frac{a_n}{(n-1)!} = 0,$$

则积分 $\int P\left(\frac{1}{x}\right)e^x dx$ 是初等函数.

求积分:

2093. $\int \left(1 - \frac{2}{x}\right)^2 e^x dx$

解 $\int \left(1 - \frac{2}{x}\right)^2 e^x dx = \int \left(1 - \frac{4}{x} + \frac{4}{x^2}\right) e^x dx$

$$= e^x - 4 \operatorname{li}(e^x) - 4 \int e^x d\left(\frac{1}{x}\right)$$

$$= e^x - 4 \operatorname{li}(e^x) - \frac{4}{x} e^x + 4 \int \frac{e^x}{x} dx$$

$$= e^x \left(1 - \frac{4}{x}\right) + C.$$

2094. $\int \left(1 - \frac{1}{x}\right) e^{-x} dx.$

解 $\int \left(1 - \frac{1}{x}\right) e^{-x} dx = -e^{-x} - \operatorname{li}(e^{-x}) + C.$

$$2095. \int \frac{e^{2x}}{x^2 - 3x + 2} dx$$

$$\begin{aligned} \text{解} \quad \int \frac{e^{2x}}{x^2 - 3x + 2} dx &= \int \frac{e^{2x}}{(x-2)(x-1)} dx \\ &= \int \frac{e^{2x}}{x-2} dx - \int \frac{e^{2x}}{x-1} dx \\ &= e^4 \int \frac{e^{2(x-2)} d(x-2)}{x-2} - e^2 \int \frac{e^{2(x-1)} d(x-1)}{x-1} \\ &= e^4 \text{li}(e^{2x-4}) - e^2 \text{li}(e^{2x-2}) + C. \end{aligned}$$

$$2096. \int \frac{xe^x}{(x+1)^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{xe^x}{(x+1)^2} dx &= - \int xe^x d\left(\frac{1}{x+1}\right) \\ &= -xe^x \frac{1}{x+1} + \int e^x dx = -\frac{xe^x}{x+1} + e^x + C \\ &= \frac{e^x}{x+1} + C. \end{aligned}$$

$$2097. \int \frac{x^4 e^{2x}}{(x-2)^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^4 e^{2x}}{(x-2)^2} dx &= \int (x^2 + 4x + 12)e^{2x} dx \\ &\quad + 32 \int \frac{e^{2x} dx}{x-2} + 16 \int \frac{e^{2x} dx}{(x-2)^2} \\ &= e^{2x} \left(\frac{x^2}{2} + \frac{3x}{2} + \frac{21}{4} \right)^* + 32e^4 \text{li}(e^{2x-4}) \\ &\quad - 16 \int e^{2x} d\left(\frac{1}{x-2}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{2x}}{2} \left(x^2 + 3x + \frac{21}{2} \right) + 32e^4 \operatorname{li}(e^{2x-4}) \\
&\quad - \frac{16e^{2x}}{x-2} + 32 \int \frac{e^{2x} dx}{x-2} \\
&= \frac{e^{2x}}{2} \left(x^2 + 3x + \frac{21}{2} - \frac{32}{x-2} \right) + 64e^4 \operatorname{li}(e^{2x-4}) + C.
\end{aligned}$$

*) 利用2066题的结果.

求含有 $\ln f(x)$, $\operatorname{arctg} f(x)$, $\operatorname{arc} \sin f(x)$, $\operatorname{arccos} f(x)$ 等函数的积分, 其中 $f(x)$ 为代数函数:

2098. $\int \ln^n x dx$ (n 为自然数).

$$\begin{aligned}
\text{解} \quad \int \ln^n x dx &= x \ln^n x - n \int \ln^{n-1} x dx \\
&= x \ln^n x - n x \ln^{n-1} x + n(n-1) \int \ln^{n-2} x dx = \dots \\
&= x \left[\ln^n x - n \ln^{n-1} x + n(n-1) \ln^{n-2} x - \dots \right. \\
&\quad \left. + (-1)^{n-1} n! \ln x + (-1)^n n! \right] + C.
\end{aligned}$$

2099. $\int x^3 \ln^3 x dx$.

$$\begin{aligned}
\text{解} \quad \int x^3 \ln^3 x dx &= \frac{1}{4} \int \ln^3 x d(x^4) \\
&= \frac{1}{4} x^4 \ln^3 x - \frac{3}{4} \int x^3 \ln^2 x dx \\
&= \frac{1}{4} x^4 \ln^3 x - \frac{3}{16} \int \ln^2 x d(x^4) \\
&= \frac{1}{4} x^4 \ln^3 x - \frac{3}{16} x^4 \ln^2 x + \frac{3}{8} \int x^3 \ln x dx.
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}x^4 \ln^3 x - \frac{3}{16}x^4 \ln^2 x + \frac{3}{32} \int \ln x d(x^4) \\
&= \frac{1}{4}x^4 \ln^3 x - \frac{3}{16}x^4 \ln^2 x + \frac{3}{32}x^4 \ln x - \frac{3}{32} \int x^3 dx \\
&= \frac{1}{4}x^4 \left(\ln^3 x - \frac{3}{4} \ln^2 x + \frac{3}{8} \ln x - \frac{3}{32} \right) + C.
\end{aligned}$$

2100. $\int \left(\frac{\ln x}{x} \right)^3 dx.$

解 $\int \left(\frac{\ln x}{x} \right)^3 dx = -\frac{1}{2} \int \ln^3 x d \left(\frac{1}{x^2} \right)$

$$\begin{aligned}
&= -\frac{1}{2x^2} \ln^3 x + \frac{3}{2} \int \frac{\ln^2 x}{x^3} dx \\
&= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4} \int \ln^2 x d \left(\frac{1}{x^2} \right) \\
&= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4x^2} \ln^2 x + \frac{3}{2} \int \frac{\ln x}{x^3} dx \\
&= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4x^2} \ln^2 x - \frac{3}{4} \int \ln x d \left(\frac{1}{x^2} \right) \\
&= -\frac{1}{2x^2} \ln^3 x - \frac{3}{4x^2} \ln^2 x - \frac{3}{4x^2} \ln x + \frac{3}{4} \int \frac{dx}{x^3} \\
&= -\frac{1}{2x^2} \left(\ln^3 x + \frac{3}{2} \ln^2 x + \frac{3}{2} \ln x + \frac{3}{4} \right) + C.
\end{aligned}$$

2101. $\int \ln[(x+a)^{x+c}(x+b)^{x+b}] \cdot \frac{dx}{(x+a)(x+b)}.$

$$\begin{aligned}
\text{解} \quad & \int \ln[(x+a)^{x+a}(x+b)^{x+b}] \cdot \frac{dx}{(x+a)(x+b)} \\
&= \int \frac{\ln(x+a)}{x+b} dx + \int \frac{\ln(x+b)}{x+a} dx \\
&= \int \ln(x+a) d[\ln(x+b)] \\
&\quad + \int \ln(x+b) d[\ln(x+a)] \\
&= \ln(x+a) \ln(x+b) - \int \ln(x+b) d[\ln(x+a)] \\
&\quad + \int \ln(x+b) d[\ln(x+a)] \\
&= \ln(x+a) \ln(x+b) + C.
\end{aligned}$$

$$2102. \int \ln^2(x + \sqrt{1+x^2}) dx.$$

$$\begin{aligned}
\text{解} \quad & \int \ln^2(x + \sqrt{1+x^2}) dx = x \ln^2(x + \sqrt{1+x^2}) \\
&\quad - 2 \int \frac{x}{\sqrt{1+x^2}} \ln(x + \sqrt{1+x^2}) dx \\
&= x \ln^2(x + \sqrt{1+x^2}) \\
&\quad - 2 \int \ln(x + \sqrt{1+x^2}) d(\sqrt{1+x^2}) \\
&= x \ln^2(x + \sqrt{1+x^2}) \\
&\quad - 2\sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2 \int dx \\
&= x \ln^2(x + \sqrt{1+x^2}) \\
&\quad - 2\sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2x + C.
\end{aligned}$$

$$2103. \int \ln(\sqrt{1-x} + \sqrt{1+x}) dx.$$

$$\text{解} \quad \int \ln(\sqrt{1-x} + \sqrt{1+x}) dx$$

$$= x \ln(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2} \int \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} dx$$

$$= x \ln(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2} \arcsin x - \frac{1}{2} x + C.$$

$$2104. \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx.$$

$$\text{解} \quad \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx = \int \ln x d\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{x \ln x}{\sqrt{1+x^2}} - \int \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{x \ln x}{\sqrt{1+x^2}} - \ln(x + \sqrt{1+x^2}) + C.$$

$$2105. \int x \operatorname{arc} \operatorname{tg}(x+1) dx.$$

$$\text{解} \quad \int x \operatorname{arc} \operatorname{tg}(x+1) dx = \frac{1}{2} \int \operatorname{arc} \operatorname{tg}(x+1) d(x^2)$$

$$= \frac{1}{2} x^2 \operatorname{arc} \operatorname{tg}(x+1) - \frac{1}{2} \int \frac{x^2}{x^2 + 2x + 2} dx$$

$$= \frac{1}{2} x^2 \operatorname{arc} \operatorname{tg}(x+1) - \frac{1}{2} \int \left(1 - \frac{2x+2}{x^2 + 2x + 2}\right) dx$$

$$= \frac{1}{2} x^2 \operatorname{arc} \operatorname{tg}(x+1) - \frac{x}{2} + \frac{1}{2} \ln(x^2 + 2x + 2) + C.$$

$$2106. \int \sqrt{x} \operatorname{arc} \operatorname{tg} \sqrt{x} dx.$$

$$\begin{aligned}
\text{解} \quad \int \sqrt{x} \operatorname{arctg} \sqrt{x} dx &= \frac{2}{3} \int \operatorname{arctg} \sqrt{x} d(x^{\frac{3}{2}}) \\
&= \frac{2}{3} x^{\frac{3}{2}} \operatorname{arctg} \sqrt{x} - \frac{1}{3} \int \frac{x}{1+x} dx \\
&= \frac{2}{3} x^{\frac{3}{2}} \operatorname{arctg} \sqrt{x} - \frac{1}{3} \int \left(1 - \frac{1}{1+x}\right) dx \\
&= \frac{2}{3} x \sqrt{x} \operatorname{arctg} \sqrt{x} - \frac{x}{3} + \frac{1}{3} \ln(1+x) + C.
\end{aligned}$$

2107. $\int x \operatorname{arcsin}(1-x) dx.$

$$\begin{aligned}
\text{解} \quad \int x \operatorname{arcsin}(1-x) dx &= \frac{1}{2} \int \operatorname{arcsin}(1-x) d(x^2) \\
&= \frac{1}{2} x^2 \operatorname{arcsin}(1-x) + \frac{1}{2} \int \frac{x^2}{\sqrt{1-(1-x)^2}} dx.
\end{aligned}$$

对于积分 $\int \frac{x^2}{\sqrt{1-(1-x)^2}} dx$, 设 $1-x=t$, 则

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-(1-x)^2}} dx &= - \int \frac{1-2t+t^2}{\sqrt{1-t^2}} dt \\
&= \int \frac{-t^2+1}{\sqrt{1-t^2}} dt - 2 \int \frac{t}{\sqrt{1-t^2}} dt + 2 \int \frac{t dt}{\sqrt{1-t^2}} \\
&= \int \sqrt{1-t^2} dt - 2 \operatorname{arcsin} t - 2 \sqrt{1-t^2} \\
&= \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \operatorname{arcsin} t - 2 \operatorname{arcsin} t \\
&\quad - 2 \sqrt{1-t^2} + C_1
\end{aligned}$$

$$= \frac{-3-x}{2} \sqrt{2x-x^2} - \frac{3}{2} \arcsin(1-x) + C_1.$$

于是,

$$\int x \arcsin(1-x) dx = \frac{2x^2-3}{4} \arcsin(1-x) - \frac{3+x}{4} \sqrt{2x-x^2} + C.$$

2108. $\int \arcsin \sqrt{x} dx.$

解 $\int \arcsin \sqrt{x} dx = x \arcsin \sqrt{x}$

$$- \frac{1}{2} \int \frac{\sqrt{x}}{\sqrt{1-x}} dx.$$

对于积分 $\int \frac{\sqrt{x}}{\sqrt{1-x}} dx$, 设 $\sqrt{x} = t$, 则 $dx = 2t dt$.

于是,

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{1-x}} dx &= 2 \int \frac{t^2}{\sqrt{1-t^2}} dt \\ &= -2 \int \sqrt{1-t^2} dt + 2 \int \frac{dt}{\sqrt{1-t^2}} \\ &= -t \sqrt{1-t^2} - \arcsin t + 2 \arcsin t + C_1 \\ &= \arcsin \sqrt{x} - \sqrt{x-x^2} + C_1. \end{aligned}$$

因而,

$$\int \arcsin \sqrt{x} dx = \left(x - \frac{1}{2}\right) \arcsin \sqrt{x}$$

$$+\frac{1}{2}\sqrt{x-x^2}+C.$$

2109. $\int x \arccos \frac{1}{x} dx.$

解
$$\begin{aligned} \int x \arccos \frac{1}{x} dx &= \frac{1}{2} \int \arccos \frac{1}{x} d(x^2) \\ &= \frac{1}{2} x^2 \arccos \frac{1}{x} - \frac{1}{2} \int \frac{|x|}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2} x^2 \arccos \frac{1}{x} - \frac{1}{2} (\operatorname{sgn} x) \sqrt{x^2-1} + C. \end{aligned}$$

2110. $\int \arcsin \frac{2\sqrt{x}}{1+x} dx.$

解
$$\begin{aligned} \int \arcsin \frac{2\sqrt{x}}{1+x} dx &= \int \arcsin \frac{2\sqrt{x}}{1+x} d(x+1) \\ &= (x+1) \arcsin \frac{2\sqrt{x}}{1+x} - \operatorname{sgn}(1-x) \int \frac{1}{\sqrt{x}} dx \\ &= (x+1) \arcsin \frac{2\sqrt{x}}{1+x} - 2\sqrt{x} \operatorname{sgn}(1-x) + C, \end{aligned}$$

其中用到了

$$\begin{aligned} \left(\arcsin \frac{2\sqrt{x}}{1+x} \right)' &= \frac{1}{(1+x)^2} \left[(1+x) \frac{1}{\sqrt{x}} - 2\sqrt{x} \right] \\ &\quad \sqrt{1 - \frac{4x}{(1+x)^2}} \\ &= \frac{1}{1+x} \cdot \frac{1-x}{\sqrt{(1-x)^2} \cdot \sqrt{x}} \end{aligned}$$

$$= \frac{1}{1+x} \cdot \operatorname{sgn}(1-x) \frac{1}{\sqrt{x}}.$$

$$2111. \int \frac{\arccos x}{(1-x^2)^{\frac{3}{2}}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{\arccos x}{(1-x^2)^{\frac{3}{2}}} dx &= \int \arccos x d\left(\frac{x}{\sqrt{1-x^2}}\right) \\ &= \frac{x \arccos x}{\sqrt{1-x^2}} + \int \frac{x dx}{1-x^2} \\ &= \frac{x \arccos x}{\sqrt{1-x^2}} - \ln \sqrt{1-x^2} + C. \end{aligned}$$

$$2112. \int \frac{x \arccos x}{(1-x^2)^{\frac{3}{2}}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x \arccos x}{(1-x^2)^{\frac{3}{2}}} dx &= \int \arccos x d\left(\frac{1}{\sqrt{1-x^2}}\right) \\ &= \frac{\arccos x}{\sqrt{1-x^2}} + \int \frac{dx}{1-x^2} \\ &= \frac{\arccos x}{\sqrt{1-x^2}} + \frac{1}{2} \ln \frac{1+x}{1-x} + C. \end{aligned}$$

$$2113. \int x \operatorname{arctg} x \ln(1+x^2) dx.$$

$$\begin{aligned} \text{解} \quad \int x \operatorname{arctg} x \ln(1+x^2) dx \\ &= \frac{1}{2} \int \operatorname{arctg} x \ln(1+x^2) d(x^2) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} x^2 \operatorname{arc\,tg} x \ln(1+x^2) \\
&\quad - \frac{1}{2} \int x^2 \left[\frac{\ln(1+x^2)}{1+x^2} + \frac{2x \operatorname{arc\,tg} x}{1+x^2} \right] dx \\
&= \frac{1}{2} x^2 \operatorname{arc\,tg} x \ln(1+x^2) - \frac{1}{2} \int \ln(1+x^2) dx \\
&\quad + \frac{1}{2} \int \frac{\ln(1+x^2)}{1+x^2} dx + \int \frac{x \operatorname{arc\,tg} x}{1+x^2} dx \\
&\quad - \int x \operatorname{arc\,tg} x dx \\
&= \frac{1}{2} x^2 \operatorname{arc\,tg} x \ln(1+x^2) \\
&\quad - \frac{1}{2} x \ln(1+x^2) + \frac{1}{2} \int \frac{2x^2 dx}{1+x^2} \\
&\quad + \frac{1}{2} \operatorname{arc\,tg} x \ln(1+x^2) - \int \frac{x \operatorname{arc\,tg} x}{1+x^2} dx \\
&\quad + \int \frac{x \operatorname{arc\,tg} x}{1+x^2} dx \\
&\quad - \frac{1}{2} x^2 \operatorname{arc\,tg} x + \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\
&= \frac{1}{2} x^2 \operatorname{arc\,tg} x \ln(1+x^2) - \frac{1}{2} x \ln(1+x^2) \\
&\quad + x - \operatorname{arc\,tg} x + \frac{1}{2} \operatorname{arc\,tg} x \ln(1+x^2) \\
&\quad - \frac{1}{2} x^2 \operatorname{arc\,tg} x + \frac{1}{2} x - \frac{1}{2} \operatorname{arc\,tg} x + C
\end{aligned}$$

$$= x - \operatorname{arc} \operatorname{tg} x + \left(\frac{1+x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{x}{2} \right) \left[\ln(1+x^2) - 1 \right] + C.$$

2114. $\int x \ln \frac{1+x}{1-x} dx.$

解 $\int x \ln \frac{1+x}{1-x} dx = \frac{1}{2} \int \ln \frac{1+x}{1-x} d(x^2)$

$$= \frac{1}{2} x^2 \ln \frac{1+x}{1-x} - \int \frac{x^2}{1-x^2} dx$$

$$= \frac{1}{2} x^2 \ln \frac{1+x}{1-x} + \int \left(1 - \frac{1}{1-x^2} \right) dx$$

$$= \frac{x^2-1}{2} \ln \frac{1+x}{1-x} + x + C.$$

2115. $\int \frac{\ln(x + \sqrt{1+x^2})}{(1+x^2)^{\frac{3}{2}}} dx.$

解 $\int \frac{\ln(x + \sqrt{1+x^2})}{(1+x^2)^{\frac{3}{2}}} dx$

$$= \int \ln(x + \sqrt{1+x^2}) d\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} - \int \frac{x}{1+x^2} dx$$

$$= \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} - \ln \sqrt{1+x^2} + C.$$

求含有双曲线函数的积分:

$$2116. \int \operatorname{sh}^2 x \operatorname{ch}^2 x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{sh}^2 x \operatorname{ch}^2 x dx &= \frac{1}{4} \int \operatorname{sh}^2 2x dx \\ &= \frac{1}{8} \int \operatorname{sh}^2 2x d(2x) \\ &= -\frac{x}{8} + \frac{\operatorname{sh} 4x}{32} + C. \end{aligned}$$

*) 利用1761题的结果。

$$2117. \int \operatorname{ch}^4 x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{ch}^4 x dx &= \int \left(\frac{1 + \operatorname{ch} 2x}{2} \right)^2 dx \\ &= \int \left(\frac{1}{4} + \frac{1}{2} \operatorname{ch} 2x + \frac{1}{4} \operatorname{ch}^2 2x \right) dx \\ &= \frac{1}{4} x + \frac{1}{4} \operatorname{sh} 2x + \frac{1}{8} \left(x + \frac{1}{4} \operatorname{sh} 4x \right) + C \\ &= \frac{3}{8} x + \frac{1}{4} \operatorname{sh} 2x + \frac{1}{32} \operatorname{sh} 4x + C. \end{aligned}$$

*) 利用1762题的结果。

$$2118. \int \operatorname{sh}^3 x dx.$$

$$\begin{aligned} \text{解} \quad \int \operatorname{sh}^3 x dx &= \int \operatorname{sh}^2 x \operatorname{sh} x dx = \int (\operatorname{ch}^2 x - 1) d(\operatorname{ch} x) \\ &= \frac{1}{3} \operatorname{ch}^3 x - \operatorname{ch} x + C. \end{aligned}$$

$$2119. \int \operatorname{sh} x \operatorname{sh} 2x \operatorname{sh} 3x dx.$$

$$\begin{aligned} \text{解} \quad & \int \operatorname{sh} x \operatorname{sh} 2x \operatorname{sh} 3x dx \\ &= \int \frac{1}{2} (\operatorname{ch} 4x - \operatorname{ch} 2x) \operatorname{sh} 2x dx \\ &= \frac{1}{2} \int \operatorname{ch} 4x \operatorname{sh} 2x dx - \frac{1}{2} \int \operatorname{ch} 2x \operatorname{sh} 2x dx \\ &= \frac{1}{4} \int (\operatorname{sh} 6x - \operatorname{sh} 2x) dx - \frac{1}{4} \int \operatorname{sh} 4x dx \\ &= \frac{1}{24} \operatorname{ch} 6x - \frac{1}{16} \operatorname{ch} 4x - \frac{1}{8} \operatorname{ch} 2x + C. \end{aligned}$$

$$2120. \int \operatorname{th} x dx.$$

$$\text{解} \quad \int \operatorname{th} x dx = \int \frac{\operatorname{sh} x}{\operatorname{ch} x} dx = \ln(\operatorname{ch} x) + C.$$

$$2121. \int \operatorname{cth}^2 x dx$$

$$\begin{aligned} \text{解} \quad & \int \operatorname{cth}^2 x dx = \int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^2 x} dx = \int \frac{1 + \operatorname{sh}^2 x}{\operatorname{sh}^2 x} dx \\ &= x - \operatorname{cth} x + C. \end{aligned}$$

$$2122. \int \sqrt{\operatorname{th} x} dx$$

$$\begin{aligned} \text{解} \quad & \int \sqrt{\operatorname{th} x} dx = \int \sqrt{\frac{e^x - e^{-x}}{e^x + e^{-x}}} dx = \int \frac{e^x - e^{-x}}{\sqrt{e^{2x} - e^{-2x}}} dx \\ &= \int \frac{e^{2x} dx}{\sqrt{e^{4x} - 1}} - \int \frac{e^{-2x} dx}{\sqrt{1 - e^{-4x}}} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{d(e^{2x})}{\sqrt{(e^{2x})^2 - 1}} + \frac{1}{2} \int \frac{d(e^{-2x})}{\sqrt{1 - (e^{-2x})^2}} \\
&= \frac{1}{2} \ln(e^{2x} + \sqrt{e^{4x} - 1}) + \frac{1}{2} \arcsin(e^{-2x}) + C.
\end{aligned}$$

2123. $\int \frac{dx}{\operatorname{sh} x + 2 \operatorname{ch} x}$

解 设 $\operatorname{th} \frac{x}{2} = t$, 则

$$\operatorname{sh} x = \frac{2t}{1-t^2}, \quad \operatorname{ch} x = \frac{1+t^2}{1-t^2},$$

$$x = \ln \frac{1+t}{1-t}, \quad dx = \frac{2}{1-t^2} dt.$$

于是,

$$\begin{aligned}
\int \frac{dx}{\operatorname{sh} x + 2 \operatorname{ch} x} &= \int \frac{dt}{t^2 + t + 1} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t+1}{\sqrt{3}} + C \\
&= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{1+2 \operatorname{th} \frac{x}{2}}{\sqrt{3}} + C.
\end{aligned}$$

2124. $\int \operatorname{sh} ax \sin bx dx$

解 $\int \operatorname{sh} ax \sin bx dx = \frac{1}{2} \int e^{ax} \sin bx dx$

$$- \frac{1}{2} \int e^{-ax} \sin bx dx$$

$$= \frac{1}{2} e^{ax} \cdot \frac{a \sin bx - b \cos bx}{a^2 + b^2}^*)$$

$$\begin{aligned}
& + \frac{1}{2} e^{-ax} \cdot \frac{a \sin bx + b \cos bx}{a^2 + b^2} + C \\
& = \frac{a \operatorname{ch} ax \cdot \sin bx - b \operatorname{sh} ax \cdot \cos bx}{a^2 + b^2} + C.
\end{aligned}$$

*) 利用1829题的结果。

2125. $\int \operatorname{sh} ax \cos bxdx.$

$$\begin{aligned}
\text{解} \quad \int \operatorname{sh} ax \cos bxdx &= \frac{1}{2} \int e^{ax} \cos bxdx \\
& - \frac{1}{2} \int e^{-ax} \cos bxdx \\
& = \frac{1}{2} e^{ax} \cdot \frac{a \cos bx + b \sin bx}{a^2 + b^2} \\
& + \frac{1}{2} e^{-ax} \cdot \frac{a \cos bx - b \sin bx}{a^2 + b^2} + C \\
& = \frac{a \operatorname{ch} ax \cdot \cos bx + b \operatorname{sh} ax \cdot \sin bx}{a^2 + b^2} + C.
\end{aligned}$$

*) 利用1828题的结果。

§6. 函数的积分法的各种例子

求积分:

2126. $\int \frac{dx}{x^6(1+x^2)}.$

$$\begin{aligned}
\text{解} \quad \int \frac{dx}{x^6(1+x^2)} &= \int \frac{(x^2+1)-x^2}{x^6(1+x^2)} dx \\
&= \int \frac{dx}{x^6} - \int \frac{dx}{x^4(1+x^2)} \\
&= -\frac{1}{5x^5} - \int \frac{(x^2+1)-x^2}{x^4(1+x^2)} dx \\
&= -\frac{1}{5x^5} - \int \frac{dx}{x^4} - \int \frac{x^2}{x^4(1+x^2)} dx \\
&= -\frac{1}{5x^5} + \frac{1}{3x^3} + \int \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx \\
&= -\frac{1}{5x^5} + \frac{1}{3x^3} - \frac{1}{x} - \text{arc tg } x + C.
\end{aligned}$$

$$2127. \int \frac{x^2 dx}{(1-x^2)^3}.$$

$$\begin{aligned}
\text{解} \quad \int \frac{x^2 dx}{(1-x^2)^3} &= \int \frac{(x^2-1)+1}{(1-x^2)^3} dx \\
&= - \int \frac{dx}{(x^2-1)^2} - \int \frac{dx}{(x^2-1)^3} \\
&= - \int \frac{dx}{(x^2-1)^2} - \left[\frac{2x}{2(-4)(x^2-1)^2} \right. \\
&\quad \left. - \frac{3}{4} \int \frac{dx}{(x^2-1)^2} \right]^{**)} \\
&= -\frac{1}{4} \int \frac{dx}{(x^2-1)^2} + \frac{x}{4(1-x^2)^2} \\
&= -\frac{1}{4} \left\{ -\frac{x}{2(x^2-1)} - \frac{1}{2} \int \frac{dx}{x^2-1} \right\} + \frac{x}{4(1-x^2)^2}
\end{aligned}$$

$$= \frac{x+x^3}{8(1-x^2)^2} - \frac{1}{16} \ln \left| \frac{1+x}{1-x} \right| + C.$$

*) 利用1921题的递推公式.

$$2128. \int \frac{dx}{1+x^4+x^8}.$$

解 因为

$$1+x^4+x^8=(x^4+1)^2-x^4=(x^4+x^2+1)(x^4-x^2+1),$$

$$x^4+x^2+1=(x^2+1)^2-x^2=(x^2-x+1)(x^2+x+1),$$

$$x^4-x^2+1=(x^2+1)^2-3x^2=(x^2+x\sqrt{3}+1)(x^2-x\sqrt{3}+1),$$

所以

$$\frac{1}{1+x^4+x^8} = \frac{1}{2} \left(\frac{x^2+1}{x^4+x^2+1} - \frac{x^2-1}{x^4-x^2+1} \right),$$

$$\frac{x^2+1}{x^4+x^2+1} = \frac{1}{2} \left(\frac{1}{x^2+x+1} + \frac{1}{x^2-x+1} \right),$$

$$\frac{x^2-1}{x^4-x^2+1} = \frac{\frac{1}{\sqrt{3}}x - \frac{1}{2}}{x^2+x\sqrt{3}+1} + \frac{\frac{1}{\sqrt{3}}x - \frac{1}{2}}{x^2-x\sqrt{3}+1}.$$

于是,

$$\begin{aligned} \int \frac{dx}{1+x^4+x^8} &= \frac{1}{4} \int \frac{dx}{x^2+x+1} + \frac{1}{4} \int \frac{dx}{x^2-x+1} \\ &+ \frac{1}{4\sqrt{3}} \int \frac{2x+\sqrt{3}}{x^2+x\sqrt{3}+1} dx \\ &- \frac{1}{4\sqrt{3}} \int \frac{2x-\sqrt{3}}{x^2-x\sqrt{3}+1} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{\frac{3}{2}}} \left[\operatorname{arctg} \left(\frac{2x+1}{\sqrt{\frac{3}{2}}} \right) + \operatorname{arctg} \left(\frac{2x-1}{\sqrt{\frac{3}{2}}} \right) \right] \\
&\quad + \frac{1}{4\sqrt{\frac{3}{2}}} \left[\ln(x^2 + x\sqrt{\frac{3}{2}} + 1) \right. \\
&\quad \left. - \ln(x^2 - x\sqrt{\frac{3}{2}} + 1) \right] + C_1 \\
&= -\frac{1}{2\sqrt{\frac{3}{2}}} \operatorname{arctg} \left(\frac{1-x^2}{x\sqrt{\frac{3}{2}}} \right) \\
&\quad + \frac{1}{4\sqrt{\frac{3}{2}}} \ln \frac{x^2 + x\sqrt{\frac{3}{2}} + 1}{x^2 - x\sqrt{\frac{3}{2}} + 1} + C.
\end{aligned}$$

2129. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$

解 设 $\sqrt[6]{x} = t$, 则

$$\sqrt{x} = t^3, \quad \sqrt[3]{x} = t^2, \quad dx = 6t^5 dt.$$

代入得

$$\begin{aligned}
\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= 6 \int \frac{t^5 dt}{t+1} \\
&= 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \\
&= 2t^3 - 3t^2 + 6t - 6\ln(1+t) + C \\
&= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1 + \sqrt[6]{x}) \\
&\quad + C \quad (x > 0).
\end{aligned}$$

2130. $\int x^2 \sqrt{\frac{x}{1-x}} dx.$

解 设 $\sqrt{\frac{1-x}{x}} = t$, 则

$$x = \frac{1}{1+t^2} \quad dx = -\frac{2t}{(1+t^2)^2} dt.$$

代入得

$$\begin{aligned} \int x^2 \sqrt{\frac{x}{1-x}} dx &= -2 \int \frac{dt}{(t^2+1)^4} \\ &= -2 \left[\frac{t}{6(t^2+1)^3} + \frac{5t}{24(t^2+1)^2} \right. \\ &\quad \left. + \frac{5t}{16(t^2+1)} + \frac{5}{16} \operatorname{arc} \operatorname{tg} t \right]^{**} + C_1 \end{aligned}$$

$$= -\frac{1}{24} (8x^2 + 10x + 15) \sqrt{x(1-x)}$$

$$- \frac{5}{8} \operatorname{arc} \operatorname{tg} \sqrt{\frac{1-x}{x}} + C_1$$

$$= -\frac{1}{24} (8x^2 + 10x + 15) \sqrt{x(1-x)}$$

$$+ \frac{5}{8} \operatorname{arc} \sin \sqrt{x} + C \quad (0 < x < 1).$$

*) 利用1921题的递推公式。

$$2131. \int \frac{x+2}{x^2 \sqrt{1-x^2}} dx.$$

解 设 $x = \sin t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则 $dx = \cos t dt$.

代入得

$$\int \frac{x+2}{x^2 \sqrt{1-x^2}} dx = \int \frac{\sin t + 2}{\sin^2 t} dt$$

$$\begin{aligned}
&= \int \frac{dt}{\sin t} + 2 \int \frac{dt}{\sin^2 t} \\
&= \ln |\csc t - \operatorname{ctg} t| - 2 \operatorname{ctg} t + C \\
&= -\ln \frac{1 + \sqrt{1-x^2}}{|x|} - \frac{2\sqrt{1-x^2}}{x} + C \quad (0 < |x| < 1).
\end{aligned}$$

2132. $\int \sqrt{\frac{x}{1-x\sqrt{x}}} dx.$

解 设 $\sqrt{1-x\sqrt{x}} = t$, 则

$$x = (1-t^2)^{\frac{2}{3}}, \quad dx = -\frac{4}{3}t(1-t^2)^{-\frac{1}{3}} dt.$$

代入得

$$\begin{aligned}
\int \sqrt{\frac{x}{1-x\sqrt{x}}} dx &= -\frac{4}{3} \int dt = -\frac{4}{3}t + C \\
&= -\frac{4}{3}\sqrt{1-x\sqrt{x}} + C \quad (0 < x < 1).
\end{aligned}$$

2133. $\int \frac{x^5 dx}{\sqrt{1+x^2}}.$

解 设 $\sqrt{1+x^2} = t$, 则 $x^2 = t^2 - 1$, $xdx = t dt$. 代入得

$$\begin{aligned}
\int \frac{x^5 dx}{\sqrt{1+x^2}} &= \int (t^2 - 1)^2 dt = \int (t^4 - 2t^2 + 1) dt \\
&= \frac{1}{5}t^5 - \frac{2}{3}t^3 + t + C \\
&= \frac{1}{15}(8 - 4x^2 + 3x^4)\sqrt{1+x^2} + C.
\end{aligned}$$

$$2134. \int \frac{dx}{\sqrt[3]{x^2(1-x)}}.$$

解 设 $\sqrt[3]{\frac{1-x}{x}} = t$, 则 $x = \frac{1}{t^3+1}$, $dx = -\frac{3t^2}{(t^3+1)^2} dt$.

代入得

$$\begin{aligned} \int \frac{dx}{\sqrt[3]{x^2(1-x)}} &= -3 \int \frac{t}{t^3+1} dt \\ &= \int \frac{dt}{t+1} - \int \frac{t+1}{t^2-t+1} dt \\ &= \ln|t+1| - \frac{1}{2} \int \frac{2t-1}{t^2-2t+1} dt - \frac{3}{2} \int \frac{dt}{t^2-t+1} \\ &= \frac{1}{2} \ln \frac{(t+1)^2}{t^2-t+1} - \sqrt{\frac{3}{3}} \operatorname{arc} \operatorname{tg} \left(\frac{2t-1}{\sqrt{\frac{3}{3}}} \right) + C, \end{aligned}$$

其中 $t = \sqrt[3]{\frac{1-x}{x}}$.

$$2135. \int \frac{dx}{x\sqrt{1+x^3+x^6}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x\sqrt{1+x^3+x^6}} &= \int \frac{dx}{x^4\sqrt{x^{-3}+x^{-3}+1}} \\ &= -\frac{1}{3} \int \frac{d\left(x^{-3} + \frac{1}{2}\right)}{\sqrt{\left(x^{-3} + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\ &= -\frac{1}{3} \ln \left| x^{-3} + \frac{1}{2} + \sqrt{x^{-6} + x^{-3} + 1} \right| + C_1 \end{aligned}$$

$$= -\frac{1}{3} \ln \left| \frac{2+x^3+2\sqrt{x^4+x^3+1}}{x^3} \right| + C.$$

注 以上实际已设 $x > 0$ 。对于 $x < 0$ ，利用1856题的方法可得同一结果。

2136. $\int \frac{dx}{x\sqrt{x^4-2x^2-1}}.$

解 $\int \frac{dx}{x\sqrt{x^4-2x^2-1}} = \int \frac{dx}{x^3\sqrt{1-2x^{-2}-x^{-4}}}$

$$= -\frac{1}{2} \int \frac{d(x^{-2}+1)}{\sqrt{2-(x^{-2}+1)^2}}$$

$$= -\frac{1}{2} \arcsin \left(\frac{x^{-2}+1}{\sqrt{2}} \right) + C_1$$

$$= -\frac{1}{2} \arcsin \left(\frac{x^2+1}{x^2\sqrt{2}} \right) + C_1$$

$$= \frac{1}{2} \arccos \left(\frac{x^2+1}{x^2\sqrt{2}} \right) + C.$$

2137. $\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} dx.$

解 $\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} dx$

$$= \int \frac{(1+\sqrt{1-x^2})(1+\sqrt{1-x^2})}{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})} dx$$

$$= \int \frac{2-x^2+2\sqrt{1-x^2}}{x^2} dx$$

$$\begin{aligned}
&= -\frac{2}{x} - x - 2 \int \sqrt{1-x^2} d\left(\frac{1}{x}\right) \\
&= -\frac{2}{x} - x - \frac{2}{x} \sqrt{1-x^2} - 2 \int \frac{dx}{\sqrt{1-x^2}} \\
&= -\frac{2+x^2}{x} - \frac{2}{x} \sqrt{1-x^2} - 2 \arcsin x + C.
\end{aligned}$$

2138. $\int \frac{(1+x)dx}{x+\sqrt{x+x^2}}$

解 $\int \frac{(1+x)dx}{x+\sqrt{x+x^2}} = \int \frac{(1+x)(x-\sqrt{x+x^2})}{(x+\sqrt{x+x^2})(x-\sqrt{x+x^2})} dx$

$$\begin{aligned}
&= \int \frac{x+x^2 - \sqrt{x+x^2} - x\sqrt{x+x^2}}{-x} dx \\
&= -x - \frac{1}{2}x^2 + \int \frac{\sqrt{1+x}}{\sqrt{x}} dx + \int \sqrt{x+x^2} dx \\
&= -x - \frac{1}{2}x^2 + 2 \int \sqrt{1+(\sqrt{x})^2} d(\sqrt{x}) \\
&\quad + \int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} d\left(x+\frac{1}{2}\right) \\
&= -x - \frac{1}{2}x^2 + \sqrt{x} \cdot \sqrt{1+x} + \ln(\sqrt{x} + \sqrt{1+x}) \\
&\quad + \frac{2x+1}{4} \sqrt{x+x^2} - \frac{1}{8} \ln\left(x+\frac{1}{2} + \sqrt{x+x^2}\right) + C_1 \\
&= -x - \frac{1}{2}x^2 + \frac{5+2x}{4} \sqrt{x+x^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \ln(2x+1+2\sqrt{x+x^2}) \\
& - \frac{1}{8} \ln\left(x+\frac{1}{2}+\sqrt{x+x^2}\right) + C_1 \\
& = -\frac{1}{2}(x+1)^2 + \frac{5+2x}{4}\sqrt{x+x^2} \\
& + \frac{3}{8} \ln\left(x+\frac{1}{2}+\sqrt{x+x^2}\right) + C,
\end{aligned}$$

其中设 $x > 0$ ，对于 $x < -1$ ，同样可获得上述结果，但要注意加对数中的绝对值。

2139. $\int \frac{\ln(1+x+x^2)}{(1+x)^2} dx.$

$$\begin{aligned}
\text{解} \quad & \int \frac{\ln(1+x+x^2)}{(1+x)^2} dx = - \int \ln(1+x+x^2) d\left(\frac{1}{1+x}\right) \\
& = -\frac{\ln(1+x+x^2)}{1+x} + \int \frac{2x+1}{(x+1)(1+x+x^2)} dx \\
& = -\frac{\ln(1+x+x^2)}{1+x} + \int \left(\frac{x+2}{1+x+x^2} - \frac{1}{1+x}\right) dx \\
& = -\frac{\ln(1+x+x^2)}{1+x} + \frac{1}{2} \int \left(\frac{2x+1}{1+x+x^2} \right. \\
& \quad \left. + \frac{3}{1+x+x^2}\right) dx - \ln|1+x| \\
& = -\frac{\ln(1+x+x^2)}{1+x} + \frac{1}{2} \ln(1+x+x^2)
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{3} \operatorname{arc\,tg} \left(\frac{2x+1}{\sqrt{3}} \right) - \ln|1+x| + C \\
& = -\frac{\ln(1+x+x^2)}{1+x} - \frac{1}{2} \ln \frac{(1+x)^2}{1+x+x^2} \\
& \quad + \sqrt{3} \operatorname{arc\,tg} \left(\frac{2x+1}{\sqrt{3}} \right) + C.
\end{aligned}$$

2140. $\int (2x+3) \operatorname{arc\,cos}(2x-3) dx,$

解 $\int (2x+3) \operatorname{arc\,cos}(2x-3) dx$

$$\begin{aligned}
& = \int \operatorname{arc\,cos}(2x-3) d(x^2+3x) \\
& = (x^2+3x) \operatorname{arc\,cos}(2x-3) + \int \frac{x^2+3x}{\sqrt{-x^2+3x-2}} dx \\
& = (x^2+3x) \operatorname{arccos}(2x-3) - \int \sqrt{-x^2+3x-2} dx \\
& \quad - 3 \int \frac{-2x+3}{\sqrt{-x^2+3x-2}} dx + 7 \int \frac{dx}{\sqrt{-x^2+3x-2}} \\
& = (x^2+3x) \operatorname{arc\,cos}(2x-3) \\
& \quad - \int \sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} d\left(x-\frac{3}{2}\right) \\
& \quad - 6\sqrt{-x^2+3x-2} + 7 \int \frac{d\left(x-\frac{3}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}}
\end{aligned}$$

$$\begin{aligned}
&= (x^2 + 3x) \arccos(2x - 3) \\
&\quad - \frac{2x - 3}{4} \sqrt{-x^2 + 3x - 2} \\
&\quad - \frac{1}{8} \arcsin(2x - 3) - 6\sqrt{-x^2 + 3x - 2} \\
&\quad - 7 \arccos(2x - 3) + C_1 \\
&= \left(x^2 + 3x - \frac{55}{8}\right) \arccos(2x - 3) \\
&\quad - \frac{2x + 21}{4} \sqrt{-x^2 + 3x - 2} + C \quad (1 < x < 2).
\end{aligned}$$

2141. $\int x \ln(4 + x^4) dx.$

解 $\int x \ln(4 + x^4) dx = \frac{1}{2} \int \ln(4 + x^4) d(x^2)$

$$\begin{aligned}
&= \frac{1}{2} x^2 \ln(4 + x^4) - 2 \int \frac{x^5}{4 + x^4} dx \\
&= \frac{1}{2} x^2 \ln(4 + x^4) - 2 \int \left(x - \frac{4x}{4 + x^4} \right) dx \\
&= \frac{1}{2} x^2 \ln(4 + x^4) - x^2 + 2 \operatorname{arctg} \left(\frac{x^2}{2} \right) + C.
\end{aligned}$$

2142. $\int \frac{\arcsin x}{x^2} \cdot \frac{1 + x^2}{\sqrt{1 - x^2}} dx.$

解 $\int \frac{\arcsin x}{x^2} \cdot \frac{1 + x^2}{\sqrt{1 - x^2}} dx$

$$\begin{aligned}
&= \int \frac{\arcsin x}{x^2 \sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx \\
&= (\operatorname{sgn} x) \int \frac{\arcsin x dx}{x^3 \sqrt{x^{-2}-1}} + \int \arcsin x d(\arcsin x) \\
&= -(\operatorname{sgn} x) \int \arcsin x d(\sqrt{x^{-2}-1}) \\
&\quad + \frac{1}{2}(\arcsin x)^2 \\
&= -(\operatorname{sgn} x) \cdot \left(\frac{\sqrt{1-x^2}}{|x|} \arcsin x - \int \frac{dx}{|x|} \right) \\
&\quad + \frac{1}{2}(\arcsin x)^2 \\
&= -\frac{\sqrt{1-x^2}}{x} \arcsin x + \int \frac{dx}{x} + \frac{1}{2}(\arcsin x)^2 \\
&= -\frac{\sqrt{1-x^2}}{x} \arcsin x + \ln|x| \\
&\quad + \frac{1}{2}(\arcsin x)^2 + C \quad (0 < |x| < 1) .
\end{aligned}$$

2143. $\int \frac{x \ln(1 + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx .$

解 $\int \frac{x \ln(1 + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$

$$\begin{aligned}
&= \int \ln(1 + \sqrt{1+x^2}) d(1 + \sqrt{1+x^2}) \\
&= (1 + \sqrt{1+x^2}) \ln(1 + \sqrt{1+x^2}) - \int \frac{x dx}{\sqrt{1+x^2}}
\end{aligned}$$

$$= (1 + \sqrt{1+x^2}) \ln(1 + \sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

2144. $\int x\sqrt{x^2+1} \ln\sqrt{x^2-1} dx.$

解 $\int x\sqrt{x^2+1} \ln\sqrt{x^2-1} dx$

$$= \frac{1}{3} \int \ln\sqrt{x^2-1} d[(x^2+1)^{\frac{3}{2}}]$$

$$= \frac{1}{3} (x^2+1)^{\frac{3}{2}} \ln\sqrt{x^2-1}$$

$$- \frac{1}{3} \int (x^2+1)^{\frac{3}{2}} \cdot \frac{x}{x^2-1} dx,$$

对于右端的积分, 设 $\sqrt{x^2+1} = t$, 则 $x^2+1 = t^2$, $xdx = tdt$. 于是,

$$- \frac{1}{3} \int (x^2+1)^{\frac{3}{2}} \frac{xdx}{x^2-1} = - \frac{1}{3} \int \frac{t^4 dt}{t^2-2}$$

$$= - \frac{1}{3} \int \left(t^2 + 2 + \frac{4}{t^2-2} \right) dt$$

$$= - \frac{1}{9} t^3 - \frac{2}{3} t - \frac{\sqrt{2}}{3} \ln \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + C$$

$$= - \frac{x^2+7}{9} \sqrt{1+x^2} - \frac{\sqrt{2}}{3} \ln \frac{\sqrt{1+x^2} - \sqrt{2}}{\sqrt{1+x^2} + \sqrt{2}} + C.$$

最后得到

$$\int x\sqrt{x^2+1} \ln\sqrt{x^2-1} dx$$

$$= \frac{1}{3} (x^2+1)^{\frac{3}{2}} \ln\sqrt{x^2-1} - \frac{x^2+7}{9} \sqrt{1+x^2}$$

$$- \frac{\sqrt{2}}{3} \ln \frac{\sqrt{1+x^2} - \sqrt{2}}{\sqrt{1+x^2} + \sqrt{2}} + C \quad (|x| \geq 1).$$

$$2145^+. \int \frac{x}{\sqrt{1-x^2}} \ln \frac{x}{\sqrt{1-x}} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{x}{\sqrt{1-x^2}} \ln \frac{x}{\sqrt{1-x}} dx \\ &= - \int \ln \frac{x}{\sqrt{1-x}} d(\sqrt{1-x^2}) \\ &= -\sqrt{1-x^2} \ln \frac{x}{\sqrt{1-x}} + \frac{1}{2} \int \frac{\sqrt{1-x^2}(2-x)}{x(1-x)} dx. \end{aligned}$$

右端的积分

$$\begin{aligned} & \int \frac{\sqrt{1-x^2}(2-x)}{x(1-x)} dx = \int \frac{(1-x^2)(2-x)}{x(1-x)\sqrt{1-x^2}} dx \\ &= \int \frac{2+x-x^2}{x\sqrt{1-x^2}} dx \\ &= 2 \int \frac{dx}{x\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{xdx}{\sqrt{1-x^2}} \\ &= -2 \int \frac{d\left(\frac{1}{x}\right)}{\sqrt{\left(\frac{1}{x}\right)^2 - 1}} + \arcsin x + \sqrt{1-x^2} \\ &= -2 \ln \left| \frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right| + \arcsin x + \sqrt{1-x^2} + C_1 \\ &= -2 \ln \frac{1 + \sqrt{1-x^2}}{x} + \arcsin x + \sqrt{1-x^2} + C_1. \end{aligned}$$

于是,

$$\int \frac{x}{\sqrt{1-x^2}} \ln \frac{x}{\sqrt{1-x}} dx$$

$$= \left(\frac{1}{2} - \ln \frac{x}{\sqrt{1-x^2}} \right) \sqrt{1-x^2} - \ln \frac{1+\sqrt{1-x^2}}{x} \\ + \frac{1}{2} \arcsin x + C \quad (0 < x < 1).$$

2146. $\int \frac{dx}{(2+\sin x)^2}$.

解 设 $\operatorname{tg} \frac{x}{2} = t$, 不妨限制 $-\pi < x < \pi$, 则

$$\sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

代入得

$$\int \frac{dx}{(2+\sin x)^2} = \frac{1}{2} \int \frac{1+t^2}{(1+t+t^2)^2} dt \\ = \frac{1}{2} \int \frac{(1+t+t^2) - \frac{1}{2}(2t+1) + \frac{1}{2}}{(1+t+t^2)^2} dt \\ = \frac{1}{2} \int \frac{dt}{1+t+t^2} - \frac{1}{4} \int \frac{(2t+1)dt}{(1+t+t^2)^2} \\ + \frac{1}{4} \int \frac{dt}{(1+t+t^2)^2} \\ = \frac{1}{\sqrt{3}} \arctg \left(\frac{2t+1}{\sqrt{3}} \right) + \frac{1}{4(1+t+t^2)} \\ + \frac{1}{4} \left[\frac{2t+1}{3(1+t+t^2)} + \frac{4}{3\sqrt{3}} \operatorname{arctg} \left(\frac{2t+1}{\sqrt{3}} \right) \right] + C_1$$

$$= \frac{4}{3\sqrt{3}} \operatorname{arc\,tg} \left(\frac{1+2\operatorname{tg}\frac{x}{2}}{\sqrt{3}} \right) + \frac{\cos x}{3(2+\sin x)}^{**} + C.$$

*) 利用1921题的递推公式.

$$**) \frac{1}{4(1+t+t^2)} + \frac{2t+1}{12(1+t+t^2)} = \frac{t+2}{6(1+t+t^2)}$$

$$\begin{aligned} & \frac{\sin\frac{x}{2} + 2\cos\frac{x}{2}}{\cos\frac{x}{2}} \\ &= \frac{1}{6} \cdot \frac{\cos\frac{x}{2}}{1 + \sin\frac{x}{2} \cos\frac{x}{2}} \\ & \quad \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} \\ &= \frac{1}{6} \cdot \frac{\frac{1}{2}\sin x + 1 + \cos x}{\frac{1}{2}\sin x + 1} = \frac{1}{6} + \frac{\cos x}{3(2+\sin x)}. \end{aligned}$$

$$2147^+. \int \frac{\sin 4x}{\sin^8 x + \cos^8 x} dx.$$

$$\text{解 } \sin^8 x + \cos^8 x = (\sin^4 x + \cos^4 x)^2 - 2\sin^4 x \cos^4 x$$

$$= [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x]^2 - \frac{1}{8} \sin^4 2x$$

$$= \left(1 - \frac{1}{2} \sin^2 2x\right)^2 - \frac{1}{8} \sin^4 2x$$

$$= \frac{1}{8} (\sin^4 2x - 8\sin^2 2x + 8).$$

$$= \frac{1}{8} (\sin^2 2x - 4 - 2\sqrt{2})(\sin^2 2x - 4 + 2\sqrt{2})$$

$$= \frac{1}{32} (\cos 4x + 7 + 4\sqrt{2}) (\cos 4x + 7 - 4\sqrt{2}).$$

于是,

$$\begin{aligned} & \int \frac{\sin 4x}{\sin^8 x + \cos^8 x} dx \\ &= 32 \cdot \frac{1}{8\sqrt{2}} \left[\int \frac{\sin 4x}{\cos 4x + 7 - 4\sqrt{2}} dx \right. \\ & \quad \left. - \int \frac{\sin 4x}{\cos 4x + 7 + 4\sqrt{2}} dx \right] \\ &= -\frac{1}{\sqrt{2}} \int \frac{d(\cos 4x + 7 - 4\sqrt{2})}{\cos 4x + 7 - 4\sqrt{2}} \\ & \quad + \frac{1}{\sqrt{2}} \int \frac{d(\cos 4x + 7 + 4\sqrt{2})}{\cos 4x + 7 + 4\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \ln \frac{\cos 4x + 7 + 4\sqrt{2}}{\cos 4x + 7 - 4\sqrt{2}} + C. \end{aligned}$$

2148. $\int \frac{dx}{\sin x \sqrt{1 + \cos x}}.$

解 设 $1 + \cos x = t^2$, 并限制 $t > 0$, 则

$$\sin x = t\sqrt{2-t^2}, \quad dx = -\frac{2}{\sqrt{2-t^2}} dt.$$

于是,

$$\begin{aligned} \int \frac{dx}{\sin x \sqrt{1 + \cos x}} &= - \int \frac{2 dt}{t^2(2-t^2)} \\ &= - \int \left(\frac{1}{t^2} + \frac{1}{2-t^2} \right) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t} - \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2}+t}{\sqrt{2}-t} + C \\
&= \frac{1}{\sqrt{1+\cos x}} - \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1+\cos x}}{\sqrt{2} - \sqrt{1+\cos x}} + C.
\end{aligned}$$

2149. $\int \frac{ax^2+b}{x^2+1} \operatorname{arc} \operatorname{tg} x dx.$

解 $\int \frac{ax^2+b}{x^2+1} \operatorname{arc} \operatorname{tg} x dx = \int \left(a - \frac{a-b}{x^2+1} \right) \operatorname{arc} \operatorname{tg} x dx$

$$\begin{aligned}
&= ax \operatorname{arc} \operatorname{tg} x - a \int \frac{x dx}{1+x^2} - \frac{a-b}{2} (\operatorname{arc} \operatorname{tg} x)^2 \\
&= a \left[x \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \ln(1+x^2) \right] - \frac{a-b}{2} (\operatorname{arc} \operatorname{tg} x)^2 + C.
\end{aligned}$$

2150. $\int \frac{ax^2+b}{x^2-1} \ln \left| \frac{x-1}{x+1} \right| dx.$

解 $\int \frac{ax^2+b}{x^2-1} \ln \left| \frac{x-1}{x+1} \right| dx$

$$\begin{aligned}
&= \int \left(a + \frac{a+b}{x^2-1} \right) \ln \left| \frac{x-1}{x+1} \right| dx \\
&= ax \ln \left| \frac{x-1}{x+1} \right| - a \int \frac{2x dx}{x^2-1} \\
&\quad + \frac{a+b}{2} \int \ln \left| \frac{x-1}{x+1} \right| d \left(\ln \left| \frac{x-1}{x+1} \right| \right) \\
&= a \left(x \ln \left| \frac{x-1}{x+1} \right| - \ln |x^2-1| \right) + \frac{a+b}{4} \ln^2 \left| \frac{x-1}{x+1} \right| + C.
\end{aligned}$$

$$2151. \int \frac{x \ln x}{(1+x^2)^2} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x \ln x}{(1+x^2)^2} dx &= -\frac{1}{2} \int \ln x d\left(\frac{1}{1+x^2}\right) \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{x(1+x^2)} \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \ln x - \frac{1}{4} \ln(1+x^2) + C \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{4} \ln \frac{x^2}{1+x^2} + C. \end{aligned}$$

$$2152. \int \frac{x \operatorname{arc} \operatorname{tg} x}{\sqrt{1+x^2}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x \operatorname{arc} \operatorname{tg} x}{\sqrt{1+x^2}} dx &= \int \operatorname{arc} \operatorname{tg} x d(\sqrt{1+x^2}) \\ &= \sqrt{1+x^2} \operatorname{arc} \operatorname{tg} x - \int \frac{dx}{\sqrt{1+x^2}} \\ &= \sqrt{1+x^2} \operatorname{arc} \operatorname{tg} x - \ln(x + \sqrt{1+x^2}) + C. \end{aligned}$$

$$2153^+. \int \frac{\sin 2x dx}{\sqrt{1+\cos^4 x}}.$$

$$\begin{aligned} \text{解} \quad \int \frac{\sin 2x dx}{\sqrt{1+\cos^4 x}} &= -\int \frac{d(1+\cos 2x)}{\sqrt{(1+\cos 2x)^2+4}} \\ &= -\ln(1+\cos 2x + \sqrt{(1+\cos 2x)^2+4}) + C_1 \\ &= -\ln(\cos^2 x + \sqrt{1+\cos^4 x}) + C. \end{aligned}$$

$$2154. \int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx.$$

$$\begin{aligned} \text{解} \quad \int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx &= - \int x^2 \arccos x d(\sqrt{1-x^2}) \\ &= -x^2 \sqrt{1-x^2} \arccos x \\ &\quad + \int \sqrt{1-x^2} \left(2x \arccos x - \frac{x^2}{\sqrt{1-x^2}} \right) dx \\ &= -x^2 \sqrt{1-x^2} \arccos x \\ &\quad - \frac{2}{3} \int \arccos x d\left[(1-x^2)^{\frac{3}{2}} \right] - \int x^2 dx \\ &= -x^2 \sqrt{1-x^2} \arccos x - \frac{2}{3} (1-x^2)^{\frac{3}{2}} \arccos x \\ &\quad - \frac{2}{3} \int (1-x^2)^{\frac{3}{2}} \cdot \frac{dx}{\sqrt{1-x^2}} - \frac{1}{3} x^3 \\ &= -x^2 \sqrt{1-x^2} \arccos x - \frac{2}{3} (1-x^2)^{\frac{3}{2}} \arccos x - \frac{2}{3} x \\ &\quad + \frac{2}{9} x^3 - \frac{1}{3} x^3 + C \\ &= -\frac{6x+x^3}{9} - \frac{2+x^2}{3} \sqrt{1-x^2} \arccos x + C. \end{aligned}$$

$$2155. \int \frac{x^4 \operatorname{arctg} x}{1+x^2} dx.$$

$$\text{解} \quad \int \frac{x^4 \operatorname{arctg} x}{1+x^2} dx = \int \left(x^2 - 1 + \frac{1}{x^2+1} \right) \operatorname{arctg} x dx$$

$$\begin{aligned}
&= \frac{1}{3} \int \operatorname{arc} \operatorname{tg} x d(x^3) - \int \operatorname{arc} \operatorname{tg} x dx \\
&\quad + \int \operatorname{arc} \operatorname{tg} x d(\operatorname{arc} \operatorname{tg} x) \\
&= \frac{1}{3} x^3 \operatorname{arc} \operatorname{tg} x - \frac{1}{3} \int \frac{x^3 dx}{1-x^2} - x \operatorname{arc} \operatorname{tg} x \\
&\quad + \int \frac{x dx}{1+x^2} + \frac{1}{2} (\operatorname{arc} \operatorname{tg} x)^2 \\
&= \frac{1}{3} x^3 \operatorname{arc} \operatorname{tg} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx - x \operatorname{arc} \operatorname{tg} x \\
&\quad + \frac{1}{2} \ln(1+x^2) + \frac{1}{2} (\operatorname{arc} \operatorname{tg} x)^2 \\
&= \frac{1}{3} x^3 \operatorname{arc} \operatorname{tg} x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1+x^2) - x \operatorname{arc} \operatorname{tg} x \\
&\quad + \frac{1}{2} \ln(1+x^2) + \frac{1}{2} (\operatorname{arc} \operatorname{tg} x)^2 + C \\
&= -\frac{1}{6} x^2 - \left(x - \frac{x^3}{3} \right) \operatorname{arc} \operatorname{tg} x \\
&\quad + \frac{1}{2} (\operatorname{arc} \operatorname{tg} x)^2 + \frac{2}{3} \ln(1+x^2) + C.
\end{aligned}$$

2156. $\int \frac{x \operatorname{arc} \operatorname{ctg} x}{(1+x^2)^2} dx.$

解 $\int \frac{x \operatorname{arc} \operatorname{ctg} x}{(1+x^2)^2} dx = -\frac{1}{2} \int \operatorname{arc} \operatorname{ctg} x d\left(\frac{1}{1+x^2}\right)$

$$= -\frac{\operatorname{arc} \operatorname{ctg} x}{2(1+x^2)} - \frac{1}{2} \int \frac{dx}{(1+x^2)^2}$$

$$= -\frac{\operatorname{arctg} x}{2(1+x^2)} - \frac{1}{2} \left[\frac{x}{2(x^2+1)} - \frac{1}{2} \operatorname{arctg} x \right]^{*}) + C$$

$$= -\frac{1-x^2}{4(1+x^2)} \operatorname{arctg} x - \frac{x}{4(1+x^2)} + C.$$

*) 利用1921题的递推公式.

$$2157^+. \int \frac{x \ln(x + \sqrt{1+x^2})}{(1-x^2)^2} dx.$$

$$\text{解} \int \frac{x \ln(x + \sqrt{1+x^2})}{(1-x^2)^2} dx$$

$$= \frac{1}{2} \int \ln(x + \sqrt{1+x^2}) d\left(\frac{1}{1-x^2}\right)$$

$$= \frac{1}{2(1-x^2)} \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \int \frac{dx}{(1-x^2)\sqrt{x^2+1}}.$$

对于右端积分设 $x = \operatorname{tg} t$, 并限制 $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$\sqrt{1+x^2} = \operatorname{sect}, \quad dx = \sec^2 t dt.$$

于是,

$$\int \frac{dx}{(1-x^2)\sqrt{1+x^2}} = \int \frac{\sec t dt}{1 - \operatorname{tg}^2 t}$$

$$= \int \frac{\cos t dt}{\cos^2 t - \sin^2 t} = \int \frac{d(\sin t)}{1 - 2\sin^2 t}$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{1 + \sqrt{2} \sin t}{1 - \sqrt{2} \sin t} \right| + C$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{1+x^2} + x\sqrt{2}}{\sqrt{1+x^2} - x\sqrt{2}} \right| + C,$$

因而,

$$\int \frac{x \ln(x + \sqrt{1+x^2})}{(1+x^2)^2} dx = \frac{\ln(x + \sqrt{1+x^2})}{2(1-x^2)} \\ + \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{1+x^2} - x\sqrt{2}}{\sqrt{1+x^2} + x\sqrt{2}} \right| + C.$$

2158. $\int \sqrt{1-x^2} \arcsin x dx.$

解 $\int \sqrt{1-x^2} \arcsin x dx = x\sqrt{1-x^2} \arcsin x$
 $- \int x \left(1 - \frac{x}{\sqrt{1-x^2}} \arcsin x\right) dx$
 $= x\sqrt{1-x^2} \arcsin x - \frac{x^2}{2}$
 $- \int \sqrt{1-x^2} \arcsin x dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx$
 $= x\sqrt{1-x^2} \arcsin x - \frac{x^2}{2} + \frac{1}{2} (\arcsin x)^2$
 $- \int \sqrt{1-x^2} \arcsin x dx,$

于是,

$$\int \sqrt{1-x^2} \arcsin x dx \\ = \frac{x}{2} \sqrt{1-x^2} \arcsin x - \frac{x^2}{4} + \frac{1}{4} (\arcsin x)^2 \\ + C \quad (|x| < 1).$$

$$2159. \int x(1+x^2) \operatorname{arc} \operatorname{ctg} x dx.$$

$$\begin{aligned} \text{解} \quad & \int x(1+x^2) \operatorname{arc} \operatorname{ctg} x dx \\ &= \frac{1}{4} \int \operatorname{arc} \operatorname{ctg} x d[(1+x^2)^2] \\ &= \frac{1}{4} (1+x^2)^2 \operatorname{arc} \operatorname{ctg} x + \frac{1}{4} \int (1+x^2) dx \\ &= \frac{1}{4} (1+x^2)^2 \operatorname{arc} \operatorname{ctg} x + \frac{x}{4} + \frac{x^3}{12} + C. \end{aligned}$$

$$2160. \int x^x(1+\ln x) dx.$$

$$\begin{aligned} \text{解} \quad & \int x^x(1+\ln x) dx = \int e^{x \ln x} (1+\ln x) dx \\ &= \int e^{x \ln x} d(x \ln x) \\ &= e^{x \ln x} + C = x^x + C \quad (x > 0). \end{aligned}$$

$$2161. \int \frac{\operatorname{arc} \operatorname{sine}^x}{e^x} dx.$$

$$\begin{aligned} \text{解} \quad & \int \frac{\operatorname{arc} \operatorname{sine}^x}{e^x} dx = - \int \operatorname{arc} \operatorname{sine}^x d(e^{-x}) \\ &= -e^{-x} \operatorname{arc} \operatorname{sine}^x + \int \frac{dx}{\sqrt{1-e^{2x}}} \\ &= -e^{-x} \operatorname{arc} \operatorname{sine}^x - \int \frac{d(e^{-x})}{\sqrt{(e^{-x})^2 - 1}} \\ &= -e^{-x} \operatorname{arc} \operatorname{sine}^x - \ln(e^{-x} + \sqrt{(e^{-2x} - 1)}) + C \end{aligned}$$

$$= x - e^{-x} \operatorname{arcsin} e^x - \ln(1 + \sqrt{1 - e^{2x}}) + C \quad (x < 0).$$

$$2162. \int \frac{\operatorname{arctg} e^{\frac{x}{2}}}{e^{\frac{x}{2}}(1+e^x)} dx.$$

$$\text{解} \quad \int \frac{\operatorname{arctg} e^{\frac{x}{2}}}{e^{\frac{x}{2}}(1+e^x)} dx = \int \left(e^{-\frac{x}{2}} - \frac{e^{\frac{x}{2}}}{1+e^x} \right) \operatorname{arctg} e^{\frac{x}{2}} dx$$

$$= -2 \int \operatorname{arctg} e^{\frac{x}{2}} d(e^{-\frac{x}{2}})$$

$$- 2 \int \operatorname{arctg} e^{\frac{x}{2}} d(\operatorname{arctg} e^{\frac{x}{2}})$$

$$= -2e^{-\frac{x}{2}} \operatorname{arctg} e^{\frac{x}{2}} + \int \frac{dx}{1+e^x} - (\operatorname{arctg} e^{\frac{x}{2}})^2$$

$$= -2e^{-\frac{x}{2}} \operatorname{arctg} e^{\frac{x}{2}} + \int \left(1 - \frac{e^x}{1+e^x} \right) dx - (\operatorname{arctg} e^{\frac{x}{2}})^2$$

$$= -2e^{-\frac{x}{2}} \operatorname{arctg} e^{\frac{x}{2}} + x - \ln(1+e^x) - (\operatorname{arctg} e^{\frac{x}{2}})^2 + C.$$

$$2163. \int \frac{dx}{(e^{x+1}+1)^2 - (e^{x-1}+1)^2}.$$

$$\text{解} \quad \int \frac{dx}{(e^{x+1}+1)^2 - (e^{x-1}+1)^2}$$

$$= \int \frac{dx}{(e^{x+1} - e^{x-1})(e^{x+1} + e^{x-1} - 2)}$$

$$= \int \frac{dx}{e^{2x}(e - e^{-1})(e + e^{-1} + 2e^{-x})}$$

$$= \int \frac{dx}{e^{2x} \cdot 2\operatorname{sh}1 \cdot (2\operatorname{ch}1 + 2e^{-x})}$$

$$\begin{aligned}
&= \int \frac{dx}{4e^x \operatorname{sh} 1 \cdot (1 + e^x \operatorname{ch} 1)} \\
&= \frac{1}{4\operatorname{sh} 1} \int \left(\frac{1}{e^x} - \frac{\operatorname{ch} 1}{1 + e^x \operatorname{ch} 1} \right) dx \\
&= -\frac{e^{-x}}{4\operatorname{sh} 1} - \frac{\operatorname{ch} 1}{4\operatorname{sh} 1} \int \left(1 - \frac{e^x \operatorname{ch} 1}{1 + e^x \operatorname{ch} 1} \right) dx \\
&= -\frac{e^{-x}}{4\operatorname{sh} 1} - \frac{\operatorname{cth} 1}{4} (x - \ln(1 + e^x \operatorname{ch} 1)) + C.
\end{aligned}$$

2164. $\int \sqrt{\operatorname{th}^2 x + 1} dx.$

$$\begin{aligned}
\text{解} \quad \int \sqrt{\operatorname{th}^2 x + 1} dx &= \int \frac{\operatorname{th}^2 x + 1}{\sqrt{\operatorname{th}^2 x + 1}} dx \\
&= \int \frac{\operatorname{sh}^2 x + \operatorname{ch}^2 x}{\operatorname{ch}^2 x \sqrt{\operatorname{th}^2 x + 1}} dx \\
&= \int \frac{2\operatorname{ch}^2 x - 1}{\sqrt{1 + \operatorname{th}^2 x}} d(\operatorname{th} x) \\
&= 2 \int \frac{\operatorname{ch}^2 x d(\operatorname{th} x)}{\sqrt{1 + \operatorname{th}^2 x}} - \int \frac{d(\operatorname{th} x)}{\sqrt{1 + \operatorname{th}^2 x}} \\
&= 2 \int \frac{dx}{\sqrt{\operatorname{th}^2 x + 1}} - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) \\
&= 2 \int \frac{\operatorname{ch} x dx}{\sqrt{\operatorname{sh}^2 x + \operatorname{ch}^2 x}} - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) \\
&= \sqrt{2} \int \frac{d(\sqrt{2} \operatorname{sh} x)}{\sqrt{1 + 2\operatorname{sh}^2 x}} - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) \\
&= \sqrt{2} \ln(\sqrt{2} \operatorname{sh} x + \sqrt{1 + 2\operatorname{sh}^2 x})
\end{aligned}$$

$$\begin{aligned}
& -\ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) + C \\
& = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{1 + \operatorname{th}^2 x} + \sqrt{2} \operatorname{th} x}{\sqrt{1 + \operatorname{th}^2 x} - \sqrt{2} \operatorname{th} x} \\
& \quad - \ln(\operatorname{th} x + \sqrt{1 + \operatorname{th}^2 x}) + C.
\end{aligned}$$

2165. $\int \frac{1 + \sin x}{1 + \cos x} \cdot e^x dx.$

解 $\int \frac{1 + \sin x}{1 + \cos x} \cdot e^x dx = \int \left(\frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) e^x dx$

$$\begin{aligned}
& = \int \frac{e^x}{2 \cos^2 \frac{x}{2}} dx + \int e^x \operatorname{tg} \frac{x}{2} dx \\
& = \int e^x d\left(\operatorname{tg} \frac{x}{2}\right) + \int \operatorname{tg} \frac{x}{2} d(e^x) \\
& = e^x \operatorname{tg} \frac{x}{2} - \int \operatorname{tg} \frac{x}{2} d e^x + \int \operatorname{tg} \frac{x}{2} d(e^x) \\
& = e^x \operatorname{tg} \frac{x}{2} + C.
\end{aligned}$$

2166. $\int |x| dx.$

解 $\int |x| dx = (\operatorname{sgn} x) \int x dx$

$$= (\operatorname{sgn} x) \frac{1}{2} x^2 + C = \frac{x|x|}{2} + C.$$

2167. $\int x|x| dx.$

$$\begin{aligned}\text{解 } \int x|x|dx &= (\operatorname{sgn}x) \int x^2 dx \\ &= (\operatorname{sgn}x) \frac{x^3}{3} + C = \frac{x^2|x|}{3} + C.\end{aligned}$$

$$2163. \int (x + |x|)^2 dx.$$

$$\begin{aligned}\text{解 } \int (x + |x|)^2 dx &= \int (x^2 + 2x|x| + x^2) dx \\ &= \frac{2x^3}{3} + \frac{2x^2|x|}{3} + C = \frac{2x^2}{3}(x + |x|) + C.\end{aligned}$$

$$2169. \int (|1+x| - |1-x|) dx.$$

$$\begin{aligned}\text{解 } \int (|1+x| - |1-x|) dx &= \int |1+x| d(1+x) + \int |1-x| d(1-x) \\ &= \frac{(1+x)|1+x|}{2} + \frac{(1-x)|1-x|}{2} + C.\end{aligned}$$

*) 利用2166题的结果.

$$2170. \int e^{-|x|} dx.$$

$$\text{解 当 } x \geq 0 \text{ 时, } \int e^{-|x|} dx = \int e^{-x} dx = -e^{-x} + C_1,$$

$$\text{当 } x < 0 \text{ 时, } \int e^{-|x|} dx = \int e^x dx = e^x + C_2.$$

由于 $e^{-|x|}$ 在 $(-\infty, +\infty)$ 上连续, 故其原函数必在 $(-\infty, +\infty)$ 上连续可微, 而且任意两个原函数之间差一常数. 今求满足 $F(0) = 0$ 的原函数 $F(x)$. 由上

述知, 必有

$$F(x) = \begin{cases} -e^{-x} + C_1, & x \geq 0, \\ e^x + C_2, & x < 0. \end{cases}$$

其中 C_1, C_2 是两个常数. 由于 $0 = F(0) = \lim_{x \rightarrow 0^-} F(x)$,

即 $0 = -1 + C_1 = 1 + C_2$, 因此 $C_1 = 1, C_2 = -1$, 从而

$$F(x) = \begin{cases} 1 - e^{-x}, & x \geq 0; \\ e^x - 1, & x < 0. \end{cases}$$

所以,

$$\int e^{-|x|} dx = \begin{cases} 1 - e^{-x} + C, & x \geq 0; \\ e^x - 1 + C, & x < 0. \end{cases}$$

2171. $\int \max(1, x^2) dx.$

解 仿上题, 当 $|x| \leq 1$ 时,

$$\int \max(1, x^2) dx = \int dx = x + C_1;$$

当 $x > 1$ 时, $\int \max(1, x^2) dx = \int x^2 dx = \frac{1}{3}x^3 + C_2,$

当 $x < -1$ 时, $\int \max(1, x^2) dx = \int x^2 dx = \frac{1}{3}x^3 + C_3.$

今求满足 $F(1) = 1$ 的原函数 $F(x)$. 由上述知

$$F(x) = \begin{cases} x + C_1, & -1 \leq x \leq 1, \\ \frac{1}{3}x^3 + C_2, & x > 1; \\ \frac{1}{3}x^3 + C_3, & x < -1. \end{cases}$$

其中 C_1, C_2, C_3 是三个常数. 由于 $1 = F(1) = \lim_{x \rightarrow 1^+} F(x)$,

即 $1 = 1 + C_1 = \frac{1}{3} + C_2$, 故 $C_1 = 0$, $C_2 = \frac{2}{3}$. 再由 $F(-1) = \lim_{x \rightarrow -1-0} F(x)$, 得 $-1 = -\frac{1}{3} + C_3$, 故 $C_3 = -\frac{2}{3}$. 由此可知

$$F(x) = \begin{cases} x, & -1 \leq x \leq 1, \\ \frac{1}{3}x^3 + \frac{2}{3}, & x > 1, \\ \frac{1}{3}x^3 - \frac{2}{3}, & x < -1. \end{cases}$$

最后得

$$\begin{aligned} & \int \max(1, x^2) dx \\ &= \begin{cases} x + C, & |x| \leq 1; \\ \frac{x^3}{3} + \frac{2}{3} \operatorname{sgn} x + C, & |x| > 1. \end{cases} \end{aligned}$$

2172. $\int \varphi(x) dx$, 其中 $\varphi(x)$ 为数 x 至其最接近的整数之距离.

解 显然 $\varphi(x)$ 在 $(-\infty, +\infty)$ 上连续, 故其原函数在 $(-\infty, +\infty)$ 上连续可微. 今求满足 $F(0) = 0$ 的原函数. 由于

$$\varphi(x) = \begin{cases} x - n, & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时,} \\ -x + n + 1, & \text{当 } n + \frac{1}{2} \leq x < n + 1 \text{ 时.} \end{cases}$$

故

$$F(x) = \begin{cases} \frac{x^2}{2} - nx + C_n, & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -\frac{x^2}{2} + (n+1)x + C'_n, & \text{当 } n + \frac{1}{2} \leq x < n+1 \text{ 时.} \end{cases}$$

其中 C_n, C'_n 是两个常数。由 $\lim_{x \rightarrow (n+\frac{1}{2})-0} F(x) = F(n + \frac{1}{2})$,

$$\text{得 } C'_n = C_n - (n + \frac{1}{2})^2.$$

故

$$F(x) = \begin{cases} \frac{x^2}{2} - nx + C_n, & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -\frac{x^2}{2} + (n+1)x - (n + \frac{1}{2})^2 + C_n, & \text{当 } n + \frac{1}{2} \leq x < n+1 \text{ 时.} \end{cases}$$

由 $\lim_{x \rightarrow (n+1)-0} F(x) = F(n+1)$

$$\text{得递推公式 } C_{n+1} = C_n + n + \frac{3}{4}.$$

显然 $0 = F(0) = C_0$ 。由此得 $C_n = \frac{1}{4}n(2n+1)$ 。

于是

$$F(x) = \begin{cases} \frac{x^2}{2} - nx + \frac{1}{4}n(2n+1) = \frac{x}{4} + \frac{1}{4}\left(x - n - \frac{1}{2}\right) \\ \quad \cdot \left[1 - 2\left(\frac{1}{2} - x + n\right)\right], & \text{当 } n \leq x < n + \frac{1}{2} \text{ 时;} \\ -\frac{x^2}{2} + (n+1)x - \frac{1}{4}(2n+1)(n+1) = \frac{x}{4} + \frac{1}{4} \\ \quad \cdot \left(x - n - \frac{1}{2}\right) \left[1 - 2\left(x - n - \frac{1}{2}\right)\right], \\ \quad \text{当 } n + \frac{1}{2} \leq x < n+1 \text{ 时.} \end{cases}$$

记 $\{x\} = x - [x]$ 表数 x 去掉其整数部分 $[x]$ 后所剩下的零头部分, 那么最后得 $F(x) = \frac{x}{4} + \frac{1}{4} \left(\{x\} - \frac{1}{2} \right) \cdot \left\{ 1 - 2 \left| \{x\} - \frac{1}{2} \right| \right\} (-\infty < x < +\infty)$.

故

$$\int \varphi(x) dx = \frac{x}{4} + \frac{1}{4} \left(\{x\} - \frac{1}{2} \right) \cdot \left\{ 1 - 2 \left| \{x\} - \frac{1}{2} \right| \right\} + C \quad (-\infty < x < +\infty).$$

2173. $\int [x] |\sin \pi x| dx \quad (x \geq 0).$

解 分别求出在区间 $(0, 1)$, $[1, 2)$, $[2, 3)$, \dots , $[[x], x)$ 上满足 $F(0) = 0$ 的原函数 $F(x)$ 的增量如下:

在 $(0, 1)$ 上, $\int 0 \cdot \sin \pi x dx = C_1, F(1) - F(0) = 0;$

在 $[1, 2)$ 上, $-\int \sin \pi x dx = \frac{1}{\pi} \cos \pi x + C_2, F(2) - F(1) = \frac{2}{\pi};$

在 $[2, 3)$ 上, $2 \int \sin \pi x dx = -\frac{2}{\pi} \cos \pi x + C_3, F(3) - F(2) = \frac{2 \cdot 2}{\pi}; \dots\dots$

在 $[[x], x)$ 上, $(-1)^{[x]} [x] \int \sin \pi x dx = (-1)^{[x]}$

$$\cdot [x] \left(-\frac{1}{\pi} \right) \cos \pi x + C_{[x]+1},$$

$$F(x) - F([x]) = \frac{(-1)^{[x]} [x]}{\pi} (\cos \pi [x] - \cos \pi x).$$

从而, 对于 $x \geq 0$, 得到

$$\begin{aligned} \int [x] |\sin \pi x| dx &= F(x) + C = (F(1) - F(0)) \\ &+ (F(2) - F(1)) + (F(3) - F(2)) + \dots \\ &+ \frac{(-1)^{[x]} [x]}{\pi} (\cos \pi [x] - \cos \pi x) + C \\ &= \frac{2}{\pi} + \frac{2 \cdot 2}{\pi} + \dots + \frac{2([x] - 1)}{\pi} \\ &+ \frac{(-1)^{[x]} [x]}{\pi} (\cos \pi [x] - \cos \pi x) + C \\ &= \frac{[x] \cdot ([x] - 1)}{\pi} + \frac{(-1)^{[x]} \cdot [x] \cdot (-1)^{[x]}}{\pi} \\ &\quad - \frac{(-1)^{[x]} \cdot [x] \cdot \cos \pi x}{\pi} + C \\ &= \frac{[x]}{\pi} ([x] - (-1)^{[x]} \cos \pi x) + C. \end{aligned}$$

2174. $\int f(x) dx$, 其中 $f(x) = \begin{cases} 1 - x^2, & \text{当 } |x| \leq 1; \\ 1 - |x|, & \text{当 } |x| > 1. \end{cases}$

解 当 $|x| \leq 1$ 时,

$$\int f(x) dx = \int (1 - x^2) dx = x - \frac{x^3}{3} + C_1;$$

$$\begin{aligned} \text{当 } x > 1 \text{ 时, } \int f(x) dx &= \int (1 - |x|) dx \\ &= x - \frac{x|x|}{2} + C_2; \end{aligned}$$

$$\begin{aligned} \text{当 } x < -1 \text{ 时, } \int f(x) dx &= \int (1 - |x|) dx \\ &= x - \frac{x|x|}{2} + C_3. \end{aligned}$$

今求满足 $F(0) = 0$ 的原函数 $F(x)$. 利用 $F(0) = 0$, $\lim_{x \rightarrow 1+0} F(x) = F(1)$, $\lim_{x \rightarrow -1-0} F(x) = F(-1)$, 仿 2171 题, 可得

$$F(x) = \begin{cases} x - \frac{x^3}{3}, & |x| \leq 1; \\ x - \frac{x|x|}{2} + \frac{1}{6}, & x > 1; \\ x - \frac{x|x|}{2} - \frac{1}{6}, & x < -1. \end{cases}$$

于是

$$\int f(x) dx = \begin{cases} x - \frac{x^3}{3} + C, & |x| \leq 1; \\ x - \frac{x|x|}{2} + \frac{1}{6} \operatorname{sgn} x + C, & |x| > 1. \end{cases}$$

2175. $\int f(x) dx$, 式中

$$f(x) = \begin{cases} 1, & \text{若 } -\infty < x < 0; \\ x+1, & \text{若 } 0 \leq x \leq 1; \\ 2x, & \text{若 } 1 < x < +\infty. \end{cases}$$

解 当 $-\infty < x < 0$ 时,

$$\int f(x)dx = \int dx = x + C_1,$$

当 $0 \leq x \leq 1$ 时,

$$\int f(x)dx = \int (x+1)dx = \frac{x^2}{2} + x + C_2,$$

当 $1 < x < +\infty$ 时,

$$\int f(x)dx = \int 2xdx = x^2 + C_3.$$

今求满足 $F(0) = 0$ 的原函数 $F(x)$. 利用 $F(0) = 0$,

$\lim_{x \rightarrow 0-0} F(x) = F(0)$, $\lim_{x \rightarrow 1+0} F(x) = F(1)$, 仿 2171 题,

可得

$$F(x) = \begin{cases} x, & \text{当 } -\infty < x < 0 \text{ 时;} \\ \frac{x^2}{2} + x, & \text{当 } 0 \leq x \leq 1 \text{ 时;} \\ x^2 + \frac{1}{2}, & \text{当 } 1 < x < +\infty \text{ 时.} \end{cases}$$

故

$$\int f(x)dx = \begin{cases} x + C, & \text{当 } -\infty < x < 0 \text{ 时;} \\ \frac{x^2}{2} + x + C, & \text{当 } 0 \leq x \leq 1 \text{ 时;} \\ x^2 + \frac{1}{2} + C, & \text{当 } 1 < x < +\infty \text{ 时.} \end{cases}$$

2176. 求 $\int x f''(x) dx$.

$$\text{解}^* \int x f''(x) dx = \int x d[f'(x)] = x f'(x)$$

$$- \int f'(x) dx = xf'(x) - f(x) + C.$$

2177. 求 $\int f'(2x) dx$.

$$\text{解}^* \int f'(2x) dx = \frac{1}{2} \int f'(2x) d(2x) = \frac{1}{2} f(2x) + C.$$

* 这里暗中分别假定了被积函数 f'' , f' 是连续的.

2178. 设 $f'(x^2) = \frac{1}{x}$ ($x > 0$), 求 $f(x)$.

$$\text{解} \text{ 由 } f'(x^2) = \frac{1}{x}, \text{ 得 } f'(x) = \frac{1}{\sqrt{x}} \quad (x > 0).$$

于是,

$$f(x) = \int f'(x) dx = \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C.$$

2179. 设 $f'(\sin^2 x) = \cos^2 x$, 求 $f(x)$.

$$\text{解} \text{ 由 } f'(\sin^2 x) = \cos^2 x = 1 - \sin^2 x \text{ 得 } f'(x) = 1 - x.$$

于是,

$$\begin{aligned} f(x) &= \int f'(x) dx = \int (1-x) dx \\ &= x - \frac{1}{2}x^2 + C \quad (|x| \leq 1). \end{aligned}$$

2180. 设 $f'(\ln x) = \begin{cases} 1, & \text{当 } 0 < x \leq 1; \\ x, & \text{当 } 1 < x < +\infty \end{cases}$

及 $f(0) = 0$, 求 $f(x)$.

解 设 $t = \ln x$, 则

$$f'(t) = \begin{cases} 1, & -\infty < t \leq 0, \\ e^t, & 0 < t < +\infty. \end{cases}$$

于是,

$$f(x) = \int f'(x) dx = \begin{cases} x + C_1, & -\infty < x \leq 0; \\ e^x + C_2, & 0 < x < +\infty, \end{cases}$$

其中 C_1, C_2 是两个常数. 由假定 $f(0) = 0$, 得 $C_1 = 0$.

再由 $f(x)$ 在 $x = 0$ 的连续性, 知 $f(0) = \lim_{x \rightarrow 0^+} f(x)$, 由

此得 $C_2 = -1$.

于是

$$f(x) = \begin{cases} x, & \text{当 } -\infty < x \leq 0 \text{ 时;} \\ e^x - 1, & \text{当 } 0 < x < +\infty \text{ 时.} \end{cases}$$

第四章 定积分

§1. 定积分作为和的极限

1° 黎曼积分的意义 若函数 $f(x)$ 在闭区间 $[a, b]$ 上有定义且 $a = x_0 < x_1 < x_2 < \dots < x_n = b$, 则数

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty, \Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i, \quad (1)$$

(式中 $x_i \leq \xi_i \leq x_{i+1}$ 及 $\Delta x_i = x_{i+1} - x_i$) 称为函数 $f(x)$ 在区间 $[a, b]$ 上的积分。

极限 (1) 存在的必要而且充分的条件为: 积分下和

$$\underline{S} = \sum_{i=0}^{n-1} m_i \Delta x_i$$

及积分上和
$$\overline{S} = \sum_{i=0}^{n-1} M_i \Delta x_i$$

当 $|\Delta x_i| \rightarrow 0$ 时有共同的极限, 其中

$$m_i = \inf_{x_i \leq x \leq x_{i+1}} f(x) \quad \text{及} \quad M_i = \sup_{x_i \leq x \leq x_{i+1}} f(x).$$

若等式 (1) 右端的极限存在, 则函数 $f(x)$ 称为在对应的区间上可积分 (常义的)。特殊情形: (a) 连续函数, (b) 具有有穷个不连续点的有界函数, (c) 单调有界的函数, —— 在任意的有穷闭区间上为可积分的。

2° 可积分的条件

$$\lim_{\max | \Delta x_i | \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0,$$

式中 ω_i 为函数 $f(x)$ 在闭区间 $[x_i, x_{i+1}]$ 上的振幅,上述等式成立为函数 $f(x)$ 于已知闭区间 $[a, b]$ 上可积分的充要条件.

2181. 把区间 $[-1, 4]$ 分为 n 个相等的子区间, 并取这些子区间中点的坐标作自变量 ξ_i 的值($i=0, 1, \dots, n-1$), 求函数 $f(x)=1+x$ 在此区间上的积分和 S_n .

解 每一个子区间的长为 $\frac{5}{n}$, 第 i 个子区间为 $(-1 +$

$\frac{5i}{n}, -1 + \frac{5i}{n} + \frac{5}{n})$, 其中点 $\xi_i = -1 + (i + \frac{1}{2})$

$\cdot \frac{5}{n}$. 于是, 所求的积分和为

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} \left\{ 1 + \left[-1 + \left(i + \frac{1}{2} \right) \frac{5}{n} \right] \right\} \cdot \frac{5}{n} \\ &= \frac{25}{n^2} \sum_{i=0}^{n-1} \left(i + \frac{1}{2} \right) = 12 \frac{1}{2}. \end{aligned}$$

2182. 设

$$(a) f(x) = x^3 \quad [-2 \leq x \leq 3];$$

$$(b) f(x) = \sqrt{x} \quad [0 \leq x \leq 1];$$

$$(B) f(x) = 2^x \quad [0 \leq x \leq 10].$$

把相应区间等分成 n 份, 求给定函数 $f(x)$ 在相应区间上的积分下和 \underline{S}_n 及积分上和 \overline{S}_n .

解 (a) 把区间 $[-2, 3]$ n 等分, 则每一个子区间的

长为 $h = \frac{5}{n}$, 且第 i 个子区间为

$$(-2 + ih, -2 + (i+1)h) \quad (i=0, 1, \dots, n-1).$$

若令 m_i 及 M_i 分别表示函数 $f(x)$ 在第 i 个子区间上的下确界及上确界, 则因 $f(x) = x^3$ 为增函数, 所以

$$m_i = (-2 + ih)^3,$$

$$M_i = [-2 + (i+1)h]^3 \quad (i=0, 1, 2, \dots, n-1).$$

于是,

$$\begin{aligned} \underline{S}_n &= \sum_{i=0}^{n-1} m_i \Delta x_i = \sum_{i=0}^{n-1} (-2 + ih)^3 h \\ &= -8nh + 12h^2 \cdot \sum_{i=0}^{n-1} i - 6h^3 \cdot \sum_{i=0}^{n-1} i^2 + h^4 \cdot \sum_{i=0}^{n-1} i^3 \\ &= -40 + \frac{12 \cdot 25n(n-1)}{2n^2} - \frac{125(2n^3 - 3n^2 + n)}{n^3} \\ &\quad + \frac{625(n^4 - 2n^3 + n^2)}{4n^4} \\ &= \frac{65}{4} - \frac{175}{2n} + \frac{125}{4n^2}; \end{aligned}$$

$$\begin{aligned} \overline{S}_n &= \sum_{i=0}^{n-1} M_i \Delta x_i = \sum_{i=0}^{n-1} (-2 + (i+1)h)^3 \\ &= \frac{65}{4} + \frac{175}{2n} + \frac{125}{4n^2}. \end{aligned}$$

(6) $h = \frac{1}{n}$,

$$m_i = \sqrt{\frac{i}{n}}.$$

$$M_i = \sqrt{\frac{i+1}{n}} \quad (i=0, 1, 2, \dots, n-1).$$

于是,

$$\underline{S}_n = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \sqrt{\frac{i}{n}} = \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{\frac{i}{n}};$$

$$\overline{S}_n = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \sqrt{\frac{i+1}{n}} = \frac{1}{n} \cdot \sum_{i=1}^n \sqrt{\frac{i}{n}}.$$

$$(B) \quad h = \frac{10}{n},$$

$$m_i = 2^{ih},$$

$$M_i = 2^{(i+1)h} \quad (i=0, 1, 2, \dots, n-1).$$

于是,

$$\underline{S}_n = \sum_{i=0}^{n-1} h 2^{ih} = \frac{h(2^{nh} - 1)}{2^h - 1} = \frac{10230}{n(2^{\frac{10}{n}} - 1)};$$

$$\overline{S}_n = \sum_{i=0}^{n-1} h 2^{(i+1)h} = \frac{h 2^h (2^{nh} - 1)}{2^h - 1}$$

$$= \frac{10230 \cdot 2^{\frac{10}{n}}}{n(2^{\frac{10}{n}} - 1)}.$$

2183. 分闭区间 $[1, 2]$ 为 n 份;使这分点的横坐标构成一等比级数^{*)},以求函数 $f(x) = x^4$ 在 $[1, 2]$ 上的积分下和.当 $n \rightarrow \infty$ 时此和的极限等于甚么?

解 设 $\sqrt[n]{2} = q$ 或 $2 = q^n$,分点为

$$1 = q^0 < q^1 < q^2 < \dots < q^n = 2.$$

由于 $f(x) = x^4$ 在 $[1, 2]$ 上为增函数,故积分下和为

$$\begin{aligned}
 \underline{S}_n &= \sum_{i=0}^{n-1} m_i \Delta x_i = \sum_{i=0}^{n-1} [(q^i)^4 \cdot (q^{i+1} - q^i)] \\
 &= (q-1) \cdot \sum_{i=0}^{n-1} (q^i)^5 = \frac{(q-1)(q^{5n} - 1)}{q^5 - 1} \\
 &= \frac{31 \cdot (\sqrt[n]{2} - 1)}{\sqrt[n]{32} - 1},
 \end{aligned}$$

且

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \underline{S}_n &= 31 \cdot \lim_{n \rightarrow \infty} \frac{\sqrt[n]{2} - 1}{\sqrt[n]{32} - 1} \\
 &= 31 \cdot \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{16} + \sqrt[n]{8} + \sqrt[n]{4} + \sqrt[n]{2} + 1} \\
 &= \frac{31}{5}.
 \end{aligned}$$

*) 原题为“使这 n 份的长构成等比级数”，现根据原题答案予以改正。

2184⁺. 从积分的定义出发，求

$$\int_0^T (v_0 + gt) dt,$$

其中 v_0 及 g 为常数。

解 $f(t) = v_0 + gt$ 在 $[0, T]$ 上为增函数 ($T > 0$)。

$$h = \frac{T}{n},$$

$$m_i = v_0 + igh,$$

$$M_i = v_0 + (i+1)gh \quad (i=0, 1, 2, \dots, n-1).$$

于是

$$\underline{S}_n = \sum_{i=0}^{n-1} (v_0 + igh) \cdot h = nv_0 h + gh^2 \sum_{i=0}^{n-1} i$$

$$= v_0 T + \frac{gT^2}{n^2} \cdot \frac{n(n-1)}{2}$$

$$= v_0 T + \frac{gT^2}{2} - \frac{gT^2}{2n},$$

$$\bar{S}_n = \sum_{i=0}^{n-1} [v_0 + (i+1)gh]h = v_0 T + \frac{gT^2}{2} - \frac{gT^2}{2n}.$$

因为

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \bar{S}_n = v_0 T + \frac{gT^2}{2},$$

所以

$$\int_0^T (v_0 + gt) dt = v_0 T + \frac{gT^2}{2}.$$

以适当的方法进行积分区间的分段，把积分看作是对应的积分和的极限，来计算定积分。

2185. $\int_{-1}^2 x^2 dx.$

解 将区间 $[-1, 2]$ n 等分，得 $h = \frac{3}{n}$ 。取

$$\xi_i = -1 + ih \quad (i=0, 1, \dots, n-1).$$

作和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} (-1 + ih)^2 h = nh - 2h^2 \sum_{i=0}^{n-1} i + h^3 \sum_{i=0}^{n-1} i^2 \\ &= 3 + \frac{9 - 9n}{2n^2}. \end{aligned}$$

于是

$$\lim_{n \rightarrow \infty} S_n = 3.$$

由于 $f(x) = x^2$ 在 $[-1, 2]$ 上连续, 故积分 $\int_{-1}^2 x^2 dx$ 是存在的, 且它与分法无关, 同时也与点的取法无关. 因此上述和的极限就是所求的积分值 (以后如无特殊情况, 不再说明), 即定积分

$$\int_{-1}^2 x^2 dx = 3.$$

2186. $\int_0^1 a^x dx \quad (a > 0).$

解 当 $a \neq 1$ 时, 将区间 $(0, 1)$ n 等分, 得 $h = \frac{1}{n}$.

取

$$\xi_i = ih \quad (i = 0, 1, \dots, n-1).$$

作和

$$S_n = \sum_{i=0}^{n-1} h a^{ih} = \frac{h(a^{nh} - 1)}{a^h - 1} = \frac{a - 1}{n(a^{\frac{1}{n}} - 1)}.$$

于是

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{a - 1}{\frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}}} = \frac{a - 1}{\ln a},$$

即

$$\int_0^1 a^x dx = \frac{a - 1}{\ln a} \quad (a \neq 1).$$

当 $a = 1$ 时, 积分显然为 1.

2187. $\int_0^{\frac{\pi}{2}} \sin x dx.$

解 将区间 $\left[0, \frac{\pi}{2}\right]$ n 等分, 得 $h = \frac{\pi}{2n}$. 取

$$\xi_i = ih \quad (i=0, 1, \dots, n-1).$$

作和

$$S_n = \sum_{i=0}^{n-1} h \sin ih.$$

由于

$$\sin ih = \frac{1}{2\sin \frac{h}{2}} \cdot \left[\cos \frac{2i-1}{2} h - \cos \frac{2i+1}{2} h \right],$$

所以

$$S_n = \frac{h}{2\sin \frac{h}{2}} \sum_{i=0}^{n-1} \left(\cos \frac{2i-1}{2} h - \cos \frac{2i+1}{2} h \right).$$

$$= \frac{h}{2\sin \frac{h}{2}} \left(\cos \frac{h}{2} - \cos \frac{2n-1}{2} h \right).$$

最后得到

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{h}{2} \cdot \lim_{n \rightarrow \infty} \left(\cos \frac{\pi}{4n} - \cos \frac{2n-1}{4n} \pi \right) \\ &= 1. \end{aligned}$$

即

$$\int_0^{\frac{\pi}{2}} \sin x dx = 1.$$

2188. $\int_0^x \cos t dt.$

解 将区间 $(0, x]$ n 等分, 得 $h = \frac{x}{n}$. 取

$$\xi_i = ih \quad (i=0, 1, \dots, n-1).$$

与2187题类似, 可得

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} h \cos ih \\ &= \lim_{n \rightarrow \infty} \frac{h}{2 \sin \frac{h}{2}} \cdot \left[\sin \frac{h}{2} + \sin \frac{2n-1}{2} h \right] \\ &= \lim_{n \rightarrow \infty} \frac{\frac{h}{2}}{\sin \frac{h}{2}} \cdot \lim_{n \rightarrow \infty} \left[\sin \frac{x}{2n} + \sin \frac{(2n-1)x}{2n} \right] \\ &= \sin x. \end{aligned}$$

即

$$\int_0^x \cos t \, dt = \sin x.$$

2189. $\int_a^b \frac{dx}{x^2} \quad (0 < a < b).$

解 将区间 (a, b) 作 n 等分, 设分点为

$$x_0 = a < x_1 < x_2 < \dots < x_n = b.$$

取 $\xi_i = \sqrt{x_i \cdot x_{i+1}}$ ($i=0, 1, 2, \dots, n-1$). 显然 $\xi_i \in (x_i, x_{i+1})$.

作和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} \frac{1}{x_i x_{i+1}} (x_{i+1} - x_i) \\ &= \sum_{i=0}^{n-1} \left(\frac{1}{x_i} - \frac{1}{x_{i+1}} \right) = \frac{1}{a} - \frac{1}{b}. \end{aligned}$$

于是

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{a} - \frac{1}{b}.$$

即

$$\int_a^b \frac{dx}{x^2} = \frac{1}{a} - \frac{1}{b}.$$

2190. $\int_a^b x^m dx$ ($0 < a < b$; $m \neq -1$).

解 选择诸分点, 使它们的横坐标构成一等比级数,
即

$$a < aq < aq^2 < \dots < aq^{i-1} < \dots < aq^{n-1} < aq^n = b,$$

其中

$$q = \sqrt[n]{\frac{b}{a}}.$$

取 $\xi_i = aq^i$ ($i=0, 1, 2, \dots, n-1$), 作和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} (aq^i)^m (aq^{i+1} - aq^i) \\ &= a^{m+1} (q-1) \cdot \sum_{i=0}^{n-1} q^{(m+1)i} \\ &= a^{m+1} (q-1) \frac{q^{n(m+1)} - 1}{q^{m+1} - 1} \\ &= (b^{m+1} - a^{m+1}) \cdot \frac{q-1}{q^{m+1} - 1}. \end{aligned}$$

由于 $\lim_{n \rightarrow \infty} q = 1$, 所以

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= (b^{m+1} - a^{m+1}) \cdot \lim_{n \rightarrow \infty} \frac{q-1}{q^{m+1} - 1} \\ &= (b^{m+1} - a^{m+1}) \lim_{n \rightarrow \infty} \frac{1}{q^m + q^{m-1} + \dots + 1} \\ &= \frac{b^{m+1} - a^{m+1}}{m+1}, \end{aligned}$$

即

$$\int_a^b x^m dx = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

2191. $\int_a^b \frac{dx}{x} \quad (0 < a < b).$

解 同2190题的分法及取法, 得和

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} (aq^i)^{-1} \cdot (aq^{i+1} - aq^i) \\ &= n(q-1) \\ &= n \left(\sqrt[n]{\frac{b}{a}} - 1 \right). \end{aligned}$$

由于 $\lim_{t \rightarrow 0} \frac{a^t - 1}{t} = \ln a \quad (a > 0)$ (可用洛比塔法则), 命 $a = \frac{b}{a}$, 而 $\frac{1}{n}$ 是趋向于 0 的变量, 应用这一极限即得

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} n \left(\sqrt[n]{\frac{b}{a}} - 1 \right) \\ &= \ln \frac{b}{a}, \end{aligned}$$

即

$$\int_a^b \frac{dx}{x} = \ln \frac{b}{a}.$$

2192. 计算布阿桑积分

$$\int_0^\pi \ln(1 - 2a \cos x + a^2) dx$$

当: (a) $|a| < 1$; (b) $|a| > 1$.

解 因为 $(1 - |\alpha|)^2 \leq 1 - 2\alpha \cos x + \alpha^2$, 所以当 $|\alpha| \neq 1$ 时, 被积函数是连续的, 于是积分就存在. 把区间 $[0, \pi]$ 分成 n 个相等部分, 即有

$$\begin{aligned} S_n &= \frac{\pi}{n} \sum_{i=1}^n \ln \left(1 - 2\alpha \cos \frac{i\pi}{n} + \alpha^2 \right) \\ &= \frac{\pi}{n} \ln \left[(1 + \alpha)^2 \prod_{i=1}^{n-1} \left(1 - 2\alpha \cos \frac{i\pi}{n} + \alpha^2 \right) \right]. \end{aligned}$$

另一方面, 我们可以证明

$$t^{2n} - 1 = (t^2 - 1) \prod_{i=1}^{n-1} \left(1 - 2t \cos \frac{i\pi}{n} + t^2 \right).$$

事实上, 方程 $t^{2n} - 1 = 0$ 共有 $2n$ 个根, 记作

$$\begin{aligned} &1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}, \varepsilon_n = -1, \\ &\bar{\varepsilon}_1, \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_{n-1}, \end{aligned}$$

其中

$$\varepsilon_i = \cos \frac{i\pi}{n} + j \sin \frac{i\pi}{n}$$

及

$$\bar{\varepsilon}_i = \cos \frac{i\pi}{n} - j \sin \frac{i\pi}{n} \quad (j = \sqrt{-1} \text{ 为虚数单位}).$$

于是

$$t^{2n} - 1 = (t+1)(t-1) \prod_{i=1}^{n-1} (t - \varepsilon_i)(t - \bar{\varepsilon}_i).$$

而

$$\begin{aligned} &\prod_{i=1}^{n-1} \left(1 - 2t \cos \frac{i\pi}{n} + t^2 \right) \\ &= \prod_{i=1}^{n-1} \left(\sin^2 \frac{i\pi}{n} + \cos^2 \frac{i\pi}{n} - 2t \cos \frac{i\pi}{n} + t^2 \right) \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^{n-1} \left[\left(t - \cos \frac{j\pi}{n} \right)^2 + \sin^2 \frac{j\pi}{n} \right] \\
&= \prod_{j=1}^{n-1} \left(t - \cos \frac{j\pi}{n} - j \sin \frac{j\pi}{n} \right) \\
&\quad \cdot \left(t - \cos \frac{j\pi}{n} + j \sin \frac{j\pi}{n} \right) \\
&= \prod_{j=1}^{n-1} (t - \varepsilon_j)(t - \bar{\varepsilon}_j) \\
&= \frac{t^{2n} - 1}{(t+1)(t-1)} \\
&= \frac{t^{2n} - 1}{t^2 - 1}.
\end{aligned}$$

即

$$t^{2n} - 1 = (t^2 - 1) \prod_{j=1}^{n-1} \left(1 - 2t \cos \frac{j\pi}{n} + t^2 \right).$$

当 $t = \alpha$ 时, 利用上式就可把 S_n 表成下面的形式

$$S_n = \frac{\pi}{n} \ln \left[\frac{\alpha+1}{\alpha-1} (\alpha^{2n} - 1) \right].$$

于是, (a) 当 $|\alpha| < 1$ 时, $\lim_{n \rightarrow \infty} S_n = 0$, 即

$$\int_0^\pi (1 - 2\alpha \cos x + \alpha^2) dx = 0.$$

(6) 当 $|\alpha| > 1$ 时, 把 S_n 改写成

$$S_n = 2\pi \ln |\alpha| + \frac{\pi}{n} \ln \left[\frac{\alpha+1}{\alpha-1} \cdot \frac{\alpha^{2n} - 1}{\alpha^{2n}} \right]$$

后, 由于 $\lim_{n \rightarrow \infty} \frac{\alpha^{2n} - 1}{\alpha^{2n}} = 1$, 从而 $\lim_{n \rightarrow \infty} S_n = 2\pi \ln |\alpha|$,

即

$$\int_0^{2\pi} \ln(1 - 2a \cos x + a^2) dx = 2\pi \ln |a|.$$

2193. 设函数 $f(x)$ 及 $\varphi(x)$ 在 $[a, b]$ 上连续, 证明

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \varphi(\theta_i) \Delta x_i = \int_a^b f(x) \varphi(x) dx.$$

其中 $x_i \leq \xi_i \leq x_{i+1}$, $x_i \leq \theta_i \leq x_{i+1}$ ($i=0, 1, \dots, n-1$) 及 $\Delta x_i = x_{i+1} - x_i$ ($x_0 = a, x_n = b$).

证 因为 $f(x)$ 及 $\varphi(x)$ 均在 $[a, b]$ 上连续, 所以它们的乘积 $f(x)\varphi(x)$ 也在 $[a, b]$ 上连续. 因此, 积分

$$\int_a^b f(x) \varphi(x) dx = \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \varphi(\xi_i) \Delta x_i \quad (1)$$

存在.

由于 $f(x)$ 在 $[a, b]$ 连续, 故有界, 即存在常数 $M > 0$, 使 $|f(x)| \leq M$ ($a \leq x \leq b$); 又由于 $\varphi(x)$ 在 $[a, b]$ 连续, 故一致连续, 因此任给 $\varepsilon > 0$, 存在 $\delta > 0$, 使当 $\max |\Delta x_i| < \delta$ 时, 恒有

$$|\varphi(\theta_i) - \varphi(\xi_i)| < \frac{\varepsilon}{M(b-a)} \quad (i=0, 1, \dots, n-1).$$

从而

$$\begin{aligned} & \left| \sum_{i=0}^{n-1} (f(\xi_i) \varphi(\theta_i) - f(\xi_i) \varphi(\xi_i)) \Delta x_i \right| \\ & \leq \sum_{i=0}^{n-1} |f(\xi_i)| \cdot |\varphi(\theta_i) - \varphi(\xi_i)| \cdot |\Delta x_i| \\ & < \sum_{i=0}^{n-1} M \cdot \frac{\varepsilon}{M(b-a)} \cdot |\Delta x_i| = \varepsilon. \end{aligned}$$

由此可知

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=1}^{n-1} [f(\xi_i)\varphi(\theta_i) - f(\xi_i)\varphi(\xi_i)] \Delta x_i = 0. \quad (2)$$

由(1)式和(2)式, 最后得到

$$\int_a^b f(x)\varphi(x)dx = \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i)\varphi(\theta_i)\Delta x_i.$$

2194. 证明不连续的函数:

$$f(x) = \operatorname{sgn} \left(\sin \frac{\pi}{x} \right)$$

于区间 $[0, 1]$ 上可积分.

证 首先注意, 函数 $f(x) = \operatorname{sgn} \left(\sin \frac{\pi}{x} \right)$ 在 $[0, 1]$ 上有界, 其不连续点是

$$0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$$

并且, $f(x)$ 在 $[0, 1]$ 的任何部分区间上的振幅 $\omega \leq 2$.

任给 $\varepsilon > 0$. 由于 $f(x)$ 在 $\left[\frac{\varepsilon}{5}, 1 \right]$ 上只有有限个间断点, 故可积. 因此存在 $\eta > 0$, 使对 $\left[\frac{\varepsilon}{5}, 1 \right]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 就有 $\sum \omega_i^* \Delta x_i < \frac{\varepsilon}{5}$. 显然, 若 $[a, \beta] \subset \left[\frac{\varepsilon}{5}, 1 \right]$, 则对于 $[a, \beta]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 也有 $\sum \omega_i^* \Delta x_i < \frac{\varepsilon}{5}$.

令 $\delta = \min\left\{\frac{\varepsilon}{5}, \eta\right\}$. 现设 $0 = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots < x_n = 1$ 是 $[0, 1]$ 的满足 $\max |\Delta x_i| < \delta$ 的任一分法. 设 $x_{i_0} \leq \frac{\varepsilon}{5} < x_{i_0+1}$.

由上述, 有 $\sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \frac{\varepsilon}{5}$. 又, 显然

$$\sum_{i=0}^{i_0} \omega_i \Delta x_i \leq 2 \sum_{i=0}^{i_0} \Delta x_i < 2 \cdot \frac{2\varepsilon}{5} = \frac{4\varepsilon}{5}.$$

故 $\sum_{i=0}^{n-1} \omega_i \Delta x_i = \sum_{i=0}^{i_0} \omega_i \Delta x_i + \sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \varepsilon$.

由此可知

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0.$$

于是, $f(x)$ 在 $(0, 1)$ 可积.

2195. 证明黎曼函数

$$\varphi(x) = \begin{cases} 0, & \text{若 } x \text{ 为无理数,} \\ \frac{1}{n}, & \text{若 } x = \frac{m}{n}, \end{cases}$$

(式中 m 及 n ($n \geq 1$) 为互质的整数) 在任何有穷的区间上可积分.

证 为简单起见, 我们只考虑闭区间 $[0, 1]$ (对于一般的有限闭区间 (a, b) , 可类似地讨论之).

命 $\lambda > 0$ 将区间 $[0, 1]$ 分成长度 $\Delta x_i < \lambda$ 的若干部分区间, 取任意的自然数 N , 将所有的部分区间分成两

类：把包含分母 $n \leq N$ 的数 $\frac{m}{n}$ 的区间列入第一类，而把不包含上述数的那些区间列入第二类。对于第一类，由于满足条件 $n \leq N$ 的数 $\frac{m}{n}$ 只有有限个，个数记为 $k = k_N$ ，所以第一类区间的个数就不大于 $2k$ ，而它们长度的总和不超出 $2k\lambda$ ；对于第二类，由于在这些区间内除含有无理数外，仅能含 $n > N$ 的有理数 $\frac{m}{n}$ ，而在这种有理点上， $\varphi\left(\frac{m}{n}\right) = \frac{1}{n} < \frac{1}{N}$ ，所以，振幅 ω_i 小于 $\frac{1}{N}$ 。

这样一来，和数 $\sum_{i=0}^{n-1} \omega_i \Delta x_i$ 就分成两部分，分别估计它们的值，即得

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i < 2k_N \lambda + \frac{1}{N}.$$

对于任意给定的 $\varepsilon > 0$ ，取定一个 $N > \frac{2}{\varepsilon}$ ，然后取 $\delta = \frac{\varepsilon}{4k_N}$ 。于是，当 $\lambda < \delta$ 时，必有

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i < \varepsilon,$$

故

$$\lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0.$$

所以函数 $\varphi(x)$ 在 $[0, 1]$ 上可积分。

2196. 证明函数

当 $x \neq 0$, $f(x) = \frac{1}{x} - \left[\frac{1}{x} \right]$ 及 $f(0) = 0$,

于闭区间 $[0, 1]$ 上可积分.

证 首先注意, 函数 $f(x)$ 在 $[0, 1]$ 上有界, 其不连续点是

$$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$$

并且, $f(x)$ 在 $[0, 1]$ 的任何部分区间上的振幅 $\omega \leq 1$.

任给 $\varepsilon > 0$. 由于 $f(x)$ 在 $\left[\frac{\varepsilon}{3}, 1 \right]$ 上只有有限个间断点, 故可积. 因此, 存在 $\eta > 0$, 使对 $\left[\frac{\varepsilon}{3}, 1 \right]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 就有 $\sum \omega_i \Delta x_i < \frac{\varepsilon}{3}$. 显然, 若 $[a, \beta] \subset \left[\frac{\varepsilon}{3}, 1 \right]$, 则对于 $[a, \beta]$ 的任何分法, 只要 $\max |\Delta x_i| < \eta$, 也有 $\sum \omega_i \Delta x_i < \frac{\varepsilon}{3}$.

令 $\delta = \min \left\{ \frac{\varepsilon}{3}, \eta \right\}$. 现设 $0 = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots < x_n = 1$ 是 $[0, 1]$ 的满足 $\max |\Delta x_i| < \delta$ 的任一分法. 设 $x_{i_0} \leq \frac{\varepsilon}{3} < x_{i_0+1}$. 由上述, 有

$$\sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \frac{\varepsilon}{3}. \text{ 又, 显然 } \sum_{i=0}^{i_0} \omega_i \Delta x_i \leq \sum_{i=0}^{i_0} \Delta x_i <$$

$$\frac{2\varepsilon}{3}. \text{ 故}$$

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i = \sum_{i=0}^{i_0} \omega_i \Delta x_i + \sum_{i=i_0+1}^{n-1} \omega_i \Delta x_i < \varepsilon.$$

于是

$$\lim_{\max \Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0.$$

由此可知, $f(x)$ 在 $[0, 1]$ 上可积.

2197. 证明迪里黑里函数

$$\chi(x) = \begin{cases} 0, & \text{若 } x \text{ 为无理数,} \\ 1, & \text{若 } x \text{ 为有理数,} \end{cases}$$

于任意区间上不可积分.

证 在任意区间 (a, b) 的任何部分区间上均有

$$\omega_i = 1,$$

所以 $\sum_{i=0}^{n-1} \omega_i \Delta x_i = b - a$, 它不趋于零. 因此函数 $\chi(x)$

在 (a, b) 上不可积分.

2198. 设函数 $f(x)$ 于 $[a, b]$ 上可积分, 且

$$f_n(x) = \sup f(x) \quad \text{当 } x_i < x \leq x_{i+1},$$

其中 $x_i = a + \frac{i}{n}(b-a)$ ($i=0, 1, \dots, n-1$).

证明 $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$.

证 $f_n(x)$ 是不超过 $n+1$ 个间断点的阶梯函数, 因此 $f_n(x)$ 在 (a, b) 上可积分, 于是

$$\left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right|$$

$$\begin{aligned}
&\leq \int_a^b |f_n(x) - f(x)| dx \\
&= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f_n(x) - f(x)| dx \\
&\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \omega_i dx = \sum_{i=0}^{n-1} \omega_i \Delta x_i \rightarrow 0 \\
&\quad (\text{当 } \max |\Delta x_i| = \frac{b-a}{n} \rightarrow 0 \text{ 时}),
\end{aligned}$$

即

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

2199. 证明: 若函数 $f(x)$ 于 $[a, b]$ 上可积分, 则有连续函数 $\varphi_n(x)$ ($n=1, 2, \dots$) 的叙列, 使

$$\int_a^c f(x) dx = \lim_{n \rightarrow \infty} \int_a^c \varphi_n(x) dx, \text{ 当 } a \leq c \leq b.$$

证 将区间 $[a, b]$ 作 n 等分, 设分点为

$$a = x_0^{(n)} < x_1^{(n)} < \dots < x_{n-1}^{(n)} < x_n^{(n)} = b,$$

即 $x_i^{(n)} = a + \frac{i}{n}(b-a)$, $i=0, 1, \dots, n$.

在 $\Delta x_i^{(n)} = [x_{i-1}^{(n)}, x_i^{(n)}]$ 上令 $\varphi_n(x)$ 为过点 $[x_{i-1}^{(n)}, f(x_{i-1}^{(n)})]$ 及 $[x_i^{(n)}, f(x_i^{(n)})]$ 的直线, 即当 $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$ 时, 令

$$\varphi_n(x) = f(x_{i-1}^{(n)}) + \frac{x - x_{i-1}^{(n)}}{x_i^{(n)} - x_{i-1}^{(n)}} [f(x_i^{(n)}) - f(x_{i-1}^{(n)})],$$

则 $\varphi_n(x)$ 是 $[a, b]$ 上的连续函数, 因此, 它是可积分的.

若令 $m_i^{(n)}$, $M_i^{(n)}$ 及 $\omega_i^{(n)}$ 分别表示函数 $f(x)$ 在 $[x_{i-1}^{(n)}, x_i^{(n)}]$ 上的下确界, 上确界及振幅, 则当 $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$ 时,

$$m_i^{(n)} \leq \varphi_n(x) \leq M_i^{(n)}, \quad m_i^{(n)} \leq f(x) \leq M_i^{(n)},$$

从而

$$|\varphi_n(x) - f(x)| \leq \omega_i^{(n)}.$$

于是, 当 $a \leq c \leq b$ 时,

$$\begin{aligned} & \left| \int_a^c f(x) dx - \int_a^c \varphi_n(x) dx \right| \\ & \leq \int_a^c |f(x) - \varphi_n(x)| dx \\ & \leq \int_a^b |f(x) - \varphi_n(x)| dx \\ & = \sum_{i=1}^n \int_{x_{i-1}^{(n)}}^{x_i^{(n)}} |f(x) - \varphi_n(x)| dx \\ & \leq \sum_{i=1}^n \omega_i^{(n)} \Delta x_i^{(n)}. \end{aligned}$$

由于 $f(x)$ 在 $[a, b]$ 上可积分, 因此,

当 $\max |\Delta x_i^{(n)}| = \frac{b-a}{n} \rightarrow 0$ 时, 必有

$$\sum_{i=1}^n \omega_i^{(n)} \Delta x_i^{(n)} \rightarrow 0.$$

由此可知

$$\int_a^c f(x) dx = \lim_{n \rightarrow \infty} \int_a^c \varphi_n(x) dx \quad (a \leq c \leq b).$$

2200. 证明: 若有界的函数 $f(x)$ 于闭区间 $[a, b]$ 上可积分,

则其绝对值 $|f(x)|$ 于 $[a, b]$ 上也可积分, 并且

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

证 对于区间 $[x_i, x_{i+1}]$ 上任意两点 x' 及 x'' , 总有

$$\left| |f(x')| - |f(x'')| \right| \leq |f(x') - f(x'')|,$$

所以函数 $|f(x)|$ 在 $[x_i, x_{i+1}]$ 上的振幅 ω_i^* 不超过 $f(x)$ 在此区间上的振幅 ω_i , 因而

$$\sum_{i=0}^{n-1} \omega_i^* \Delta x_i \leq \sum_{i=0}^{n-1} \omega_i \Delta x_i \rightarrow 0,$$

即 $|f(x)|$ 在 $[a, b]$ 上可积分.

其次, 因为

$$-|f(x)| \leq f(x) \leq |f(x)|,$$

所以

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx,$$

即

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

2201. 考函数 $f(x)$ 于闭区间 $[a, b]$ 上绝对可积 (即积分 $\int_a^b |f(x)| dx$ 存在). 这个函数在 $[a, b]$ 上是否为可积分的函数?

解 一般地说, 不. 例如函数

$$f(x) = \begin{cases} 1, & \text{若 } x \text{ 为有理数,} \\ -1, & \text{若 } x \text{ 为无理数.} \end{cases}$$

$|f(x)| = 1$ ，它在 $[a, b]$ 上连续，因此它在 (a, b) 上可积。但对于函数 $f(x)$ 而言，在 $[a, b]$ 的任一部分区间上 $\omega_i = 2$ ，所以

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i = 2(b-a).$$

它不趋向于零，于是函数 $f(x)$ 在 (a, b) 上不可积分。

2202. 设函数 $\varphi(x)$ 于闭区间 $[A, B]$ 上有定义并连续，函数 $f(x)$ 于 (a, b) 上可积分，并且当 $a \leq x \leq b$ 时 $A \leq f(x) \leq B$ 。证明函数 $\varphi[f(x)]$ 于 $[a, b]$ 上可积分。

证 任给 $\varepsilon > 0$ 。根据函数 $\varphi(x)$ 在 $[A, B]$ 的一致连续性，存在 $\eta > 0$ ，使得在 $[A, B]$ 中长度小于 η 的任一闭区间上，函数 φ 的振幅都小于 $\frac{\varepsilon}{2(b-a)}$ 。用 Ω 表 $\varphi(x)$ 在 $[A, B]$ 上的振幅。由 $f(x)$ 在 (a, b) 的可积性，知必有 $\delta > 0$ 存在，使对 $[a, b]$ 的任一分法，只要 $\max |\Delta x_i| < \delta$ ，就有 $\sum_{i=0}^{n-1} \omega_i(f) \Delta x_i < \frac{\eta \varepsilon}{2\Omega}$ 。（ $\omega_i(f)$ 表 $f(x)$ 在 (x_i, x_{i+1}) 上的振幅）。

下证对 (a, b) 的任何分法，只要 $\max |\Delta x_i| < \delta$ ，就有

$$\sum_{i=0}^{n-1} \omega_i(\varphi(f)) \Delta x_i < \varepsilon.$$

事实上，将诸区间 $[x_i, x_{i+1}]$ 分成两组，第一组是满足 $\omega_i(f) < \eta$ 的（其下标以“ i' ”记之），第二组是满足 $\omega_i(f) \geq \eta$ 的（下标以“ i'' ”记之）。

于是,

$$\begin{aligned} & \sum_{i=0}^{n-1} \omega_i(\varphi(f)) \Delta x_i \\ &= \sum_r \omega_r(\varphi(f)) \Delta x_r + \sum_{r'} \omega_{r'}(\varphi(f)) \Delta x_{r'} \\ &< \frac{\varepsilon}{2(b-a)} \sum_r \Delta x_r + \Omega \sum_{r'} \Delta x_{r'}, \end{aligned}$$

但

$$\begin{aligned} & \frac{\eta\varepsilon}{2\Omega} > \sum_{i=0}^{n-1} \omega_i(f) \Delta x_i \\ &= \sum_r \omega_r(f) \Delta x_r + \sum_{r'} \omega_{r'}(f) \Delta x_{r'} \\ &\geq \sum_{r'} \omega_{r'}(f) \Delta x_{r'} \geq \eta \sum_{r'} \Delta x_{r'}, \end{aligned}$$

于是

$$\begin{aligned} & \sum_{i=0}^{n-1} \omega_i(\varphi(f)) \Delta x_i \\ &< \frac{\varepsilon}{2(b-a)} \cdot (b-a) + \Omega \cdot \frac{\varepsilon}{2\Omega} = \varepsilon. \end{aligned}$$

由此可知, $\varphi[f(x)]$ 在 (a, b) 上可积.

2203. 若函数 $f(x)$ 及 $\varphi(x)$ 可积分, 则函数 $f(\varphi(x))$ 是否也必定可积分?

解 未必. 例如函数

$$f(x) = \begin{cases} 0, & \text{若 } x=0, \\ 1 & \text{若 } x \neq 0, \end{cases}$$

及 $\varphi(x)$ 为黎曼函数 (参阅 2195 题).

它们在任何有穷的区间上均可积（前者不连续点仅为原点一个，且是有界函数，因而是可积分的）。

但 $f(\varphi(x)) = x(x)$ ，利用 2197 题的结果得知它在任何有穷的区间上不可积分。

2204. 设函数 $f(x)$ 于闭区间 $[A, B]$ 上可积分，证明函数 $f(x)$ 有积分的连续性，即是说

$$\lim_{h \rightarrow 0} \int_a^{a+h} |f(x+h) - f(x)| dx = 0,$$

式中 $(a, b) \subset (A, B)$ 。

证 方法一：

不妨设 $A < a, b < B$ 。由于 $f(x)$ 在 (A, B) 可积，故任给 $\varepsilon > 0$ ，存在 $\eta > 0$ ，使对 (A, B) 的任何分法，只要 $\max |\Delta x_i| < \eta$ ，就恒有

$$\sum_i \omega_i \Delta x_i < \varepsilon,$$

显然，对 (A, B) 的任一子区间 (A', B') 的任何分法，只要 $\max |\Delta x_{i'}| < \eta$ ，也有

$$\sum_{i'} \omega_{i'} \Delta x_{i'} < \varepsilon. \quad (1)$$

今设 $0 < h < \delta = \min \left\{ \frac{\eta}{2}, \frac{B-b}{3} \right\}$ ，则对于 h ，

存在正整数 $n = n(h)$ ，使有

$$a + (2n-2)h < b \leq a + 2nh < a + (2n+1)h < B.$$

用 ω_i 表 $f(x)$ 在 $(a+ih, a+(i+2)h)$ 上的振幅，则

$$\int_a^b |f(x+h) - f(x)| dx$$

$$\begin{aligned}
&\leq \int_a^{a+2nh} |f(x+h) - f(x)| dx \\
&= \sum_{i=0}^{2n-1} \int_{a+ih}^{a+(i+1)h} |f(x+h) - f(x)| dx \leq \sum_{i=0}^{2n-1} \omega_i h \\
&= \frac{1}{2} \sum_{i=0}^{n-1} \omega_{2i} 2h + \frac{1}{2} \sum_{i=0}^{n-1} \omega_{2i+1} 2h.
\end{aligned}$$

显然, $\sum_{i=0}^{n-1} \omega_{2i} \cdot 2h$ 是对于区间 $[a, a+2nh]$ 的分法 $a < a+2h < a+4h < \dots < a+2nh$ 所作的 (1) 式中的和, 而 $\sum_{i=0}^{n-1} \omega_{2i+1} 2h$ 是对于区间 $[a+h, a+(2n+1)h]$

的分法

$a+h < a+3h < a+5h < \dots < a+(2n+1)h$ 所作的 (1) 式中的和. 故

$$\sum_{i=0}^{n-1} \omega_{2i} 2h < \varepsilon, \quad \sum_{i=0}^{n-1} \omega_{2i+1} 2h < \varepsilon.$$

从而

$$\int_a^b |f(x+h) - f(x)| dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

由此可知

$$\lim_{h \rightarrow 0^+} \int_a^b |f(x+h) - f(x)| dx = 0.$$

同理可证

$$\lim_{h \rightarrow 0^-} \int_a^b |f(x+h) - f(x)| dx = 0.$$

于是, 得

$$\lim_{h \rightarrow 0} \int_a^b |f(x+h) - f(x)| dx = 0.$$

方法二:

由2199题的结果可知: 对于任意给定的 $\varepsilon > 0$, 由于 $f(x)$ 在 $[A, B]$ 上可积, 故存在 $[A, B]$ 上的连续函数 $\varphi(x)$, 使

$$\int_A^B |f(x) - \varphi(x)| dx < \frac{\varepsilon}{4}.$$

由于 $\varphi(x)$ 在 $[A, B]$ 上一致连续, 故存在 $\delta > 0$, 使当 $|x' - x''| < \delta$ ($x' \in [A, B]$, $x'' \in [A, B]$) 时, 恒有

$$|\varphi(x') - \varphi(x'')| < \frac{\varepsilon}{2(b-a)}.$$

于是, 当 $|h| < \delta$ 时,

$$\begin{aligned} & \int_a^b |f(x+h) - f(x)| dx \\ & \leq \int_a^b |f(x+h) - \varphi(x+h)| dx \\ & \quad + \int_a^b |\varphi(x+h) - \varphi(x)| dx \\ & \quad + \int_a^b |f(x) - \varphi(x)| dx \\ & \leq 2 \int_A^B |f(x) - \varphi(x)| dx \\ & \quad + \int_a^b |\varphi(x+h) - \varphi(x)| dx \end{aligned}$$

$$< 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon.$$

故

$$\lim_{h \rightarrow 0} \int_a^b |f(x+h) - f(x)| dx = 0.$$

2205. 设函数 $f(x)$ 于闭区间 $[a, b]$ 上可积分, 证明等式

$$\int_a^b f^2(x) dx = 0$$

当而且仅当对属于闭区间 $[a, b]$ 内函数 $f(x)$ 连续的一切点有 $f(x) = 0$ 时方成立.

证 先证必要性:

采用反证法. 设 $f(x)$ 在点 x_0 连续, 但 $f(x_0) \neq 0$, 则存在 $\delta > 0$, $[x_0 - \delta, x_0 + \delta] \subset (a, b)$, 使当 $|x - x_0| \leq \delta$ 时

$$|f(x)| > \frac{|f(x_0)|}{2}$$

从而

$$\begin{aligned} \int_a^b f^2(x) dx &\geq \int_{x_0 - \delta}^{x_0 + \delta} f^2(x) dx > \frac{f^2(x_0)}{4} \cdot 2\delta \\ &= \frac{\delta \cdot f^2(x_0)}{2} > 0. \end{aligned}$$

这与假设 $\int_a^b f^2(x) dx = 0$ 矛盾.

再证充分性:

也即要证: $f(x)$ 在 $[a, b]$ 上可积条件下, 假设 $f(x)$ 在一切连续点 x_0 上均有 $f(x_0) = 0$, 则必有

$$\int_a^b f^2(x) dx = 0.$$

证明分两个部分。第一，首先要指出当 $f(x)$ 在 $[a, b]$ 上可积时， $f(x)$ 的连续点在 $[a, b]$ 中必定是稠密的。此处所谓“稠密”性是指：对于任意区间 $[\alpha, \beta] \subset [a, b]$ 总存在一点 $x_0 \in [\alpha, \beta]$ ，使 $f(x)$ 在 x_0 连续。第二，利用假设，并借助于稠密性，可证得充分性。现在先证第二部分，如下：由 $f(x)$ 在 $[a, b]$ 上的全体连续点 X 的稠密性以及当 $x_0 \in X$ 时有 $f(x_0) = 0$ 的假设。便知，对于区间 $[a, b]$ 的任一分法，均可适当地取 $x_i \leq \xi_i \leq x_{i+1}$ ，使 $f(\xi_i) = 0$ 。

从而积分和 $\sum_{i=0}^{n-1} f^2(\xi_i) \Delta x_i = 0$ 。由此，再注意到 $f^2(x)$ 在 $[a, b]$ 的可积性，便有

$$\int_a^b f^2(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f^2(\xi_i) \Delta x_i = 0.$$

如今再补证第一部分：应当首先指明，若 $f(x)$ 在 $[a, \beta]$ 上可积，则对任给的 $\varepsilon > 0$ ，总存在 $[a, \beta]$ 的子区间 $[a', \beta']$ 使得振幅

$$\omega(a', \beta') < \varepsilon.$$

事实上，如果上述结论不成立，则存在一个 $\varepsilon_0 > 0$ ，使对于 $[a, \beta]$ 的任意分法，有

$$\sum \omega_i \Delta x_i \geq \varepsilon_0 \sum \Delta x = \varepsilon_0 (\beta - a) > 0,$$

这与 $f(x)$ 在 $[a, \beta]$ 可积矛盾，因此，结论为真。

今取 (α, β) 为 (a_1, b_1) . 由于 $f(x)$ 在 $\left[a_1 + \frac{b_1 - a_1}{4}, b_1 - \frac{b_1 - a_1}{4} \right]$ 上可积, 故存在区间 $(a_2, b_2) \subset \left[a_1 + \frac{b_1 - a_1}{4}, b_1 - \frac{b_1 - a_1}{4} \right] \subset (a_1, b_1)$, 使

$$\omega(a_2, b_2) < \frac{1}{2}.$$

同样, 存在区间 $(a_3, b_3) \subset \left[a_2 + \frac{b_2 - a_2}{4}, b_2 - \frac{b_2 - a_2}{4} \right] \subset (a_2, b_2)$, 使

$$\omega(a_3, b_3) < \frac{1}{3}.$$

这样继续下去, 得一串闭区间 (a_n, b_n) ($n=1, 2, 3, \dots$), 满足

$$a = a_1 < a_2 < \dots < a_n < \dots < b_n < \dots < b_2 < b_1 = \beta,$$

并且 $b_n - a_n \leq \frac{\beta - \alpha}{2^{n-1}} \rightarrow 0$, $\omega(a_n, b_n) < \frac{1}{n}$ ($n=1, 2, 3, \dots$).

由区间套定理, 诸 (a_n, b_n) 具有唯一的公共点 c . 显然 $a_n < c < b_n$ ($n=1, 2, 3, \dots$). 下证 $f(x)$ 在点 c 连续.

任给 $\varepsilon > 0$, 取正整数 n_0 使 $n_0 > \frac{1}{\varepsilon}$. 再取 $\delta > 0$ 使 $(c - \delta, c + \delta) \subset (a_{n_0}, b_{n_0})$. 于是, 当 $|x - c| < \delta$ 时, 必有 $|f(x) - f(c)| \leq \omega(a_{n_0}, b_{n_0}) < \frac{1}{n_0} < \varepsilon$.

故 $f(x)$ 在点 $x=c$ 连续。到此，充分性证毕。

§2. 利用不定积分计算定积分的方法

1° 牛顿—莱布尼兹公式 若函数 $f(x)$ 于闭区间 $[a, b]$ 上有定义而且连续， $F(x)$ 为它的原函数(即 $F'(x)=f(x)$)，则

$$\int_a^b f(x)dx = F(b) - F(a) = F(x) \Big|_a^b.$$

定积分 $\int_a^b f(x)dx$ 的几何意义表示由曲线 $y=f(x)$ ， OX 轴及垂直于 OX 轴的二直线 $x=a$ 和 $x=b$ 四者所围成的代数面积 S (图4.1)。

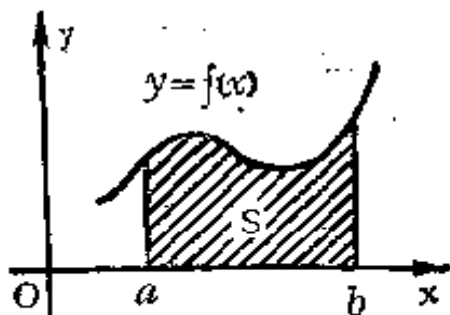


图 4.1

2° 部分积分法 若函数 $f(x)$ 和 $g(x)$ 于闭区间 $[a, b]$ 上连续并有连续导数 $f'(x)$ 和 $g'(x)$ ，则

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x)dx.$$

3° 变数代换 若：(1) 函数 $f(x)$ 于闭区间 $[a, b]$ 内连续，(2) 函数 $\varphi(t)$ 及其导数 $\varphi'(t)$ 皆于闭区间 $[\alpha, \beta]$ 上连续，其中 $a=\varphi(\alpha)$ ； $b=\varphi(\beta)$ ；(3) 复合函数 $f[\varphi(t)]$ 于闭区间 $[\alpha, \beta]$ 上有定义并连续，则

※ 本节个别题是收敛的广义积分，仍按此公式计算。——题解编者注。

$$\int_a^b f(x) dx = \int_a^b f(\varphi(t)) \varphi'(t) dt.$$

利用牛顿—莱布尼兹公式，求下列定积分并绘出对应的曲边图形面积：

$$2206. \int_{-1}^8 \sqrt[3]{x} dx.$$

$$\text{解} \quad \int_{-1}^8 \sqrt[3]{x} dx = \frac{3}{4} x^{\frac{4}{3}} \Big|_{-1}^8 = 11\frac{1}{4} \quad (\text{图 4.2}).$$

$$2207. \int_0^{\pi} \sin x dx$$

$$\text{解} \quad \int_0^{\pi} \sin x dx = -\cos x \Big|_0^{\pi} = 2 \quad (\text{图 4.3}).$$



图 4.2



图 4.3

$$2208. \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$$

$$\text{解} \quad \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \arctg x \Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{\pi}{6} \quad (\text{图 4.4}).$$

$$2209. \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$$

$$\text{解} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3} \quad (\text{图 4.5}).$$

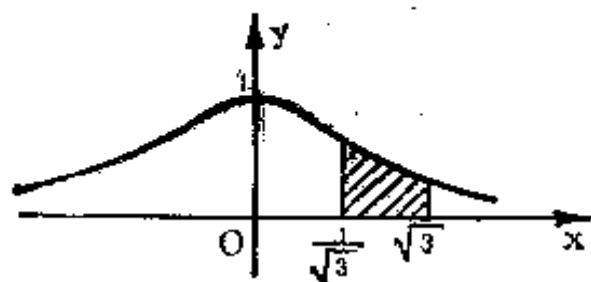


图 4.4

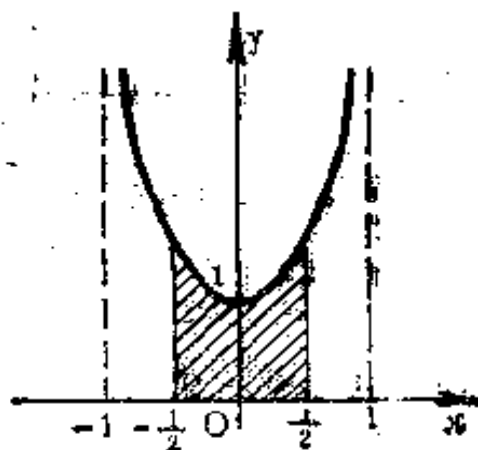


图 4.5

2210. $\int_{sh1}^{sh2} \frac{dx}{\sqrt{1+x^2}}$

解 $\int_{sh1}^{sh2} \frac{dx}{\sqrt{1+x^2}} = \ln(x + \sqrt{1+x^2}) \Big|_{sh1}^{sh2}$

$\ln(x + \sqrt{1+x^2}) \Big|_{sh1}^{sh2} = \ln \frac{sh2 + \sqrt{1+sh2^2}}{sh1 + \sqrt{1+sh1^2}}$ (图 4.6)

$$2212. \int_{-1}^1 \frac{dx}{x^2 - 2x \cos \alpha + 1} \quad (0 < \alpha < \pi).$$

$$\begin{aligned} \text{解} \quad & \int_{-1}^1 \frac{dx}{x^2 - 2x \cos \alpha + 1} \\ &= \frac{1}{\sin \alpha} \operatorname{arc} \operatorname{tg} \frac{x - \cos \alpha}{\sin \alpha} \Big|_{-1}^1 \\ &= \frac{1}{\sin \alpha} \left[\operatorname{arc} \operatorname{tg} \left(\operatorname{tg} \frac{\alpha}{2} \right) + \operatorname{arc} \operatorname{tg} \left(\operatorname{ctg} \frac{\alpha}{2} \right) \right] \\ &= \frac{1}{\sin \alpha} \left\{ \frac{\alpha}{2} + \operatorname{arc} \operatorname{tg} \left[\operatorname{tg} \left(\frac{\pi}{2} - \frac{\alpha}{2} \right) \right] \right\} \\ &= \frac{\pi}{2 \sin \alpha} \quad (\text{图 4.8}). \end{aligned}$$

注 以下图形从略。

$$2213. \int_0^{2\pi} \frac{dx}{1 + e \cos x} \quad (0 \leq e < 1).$$

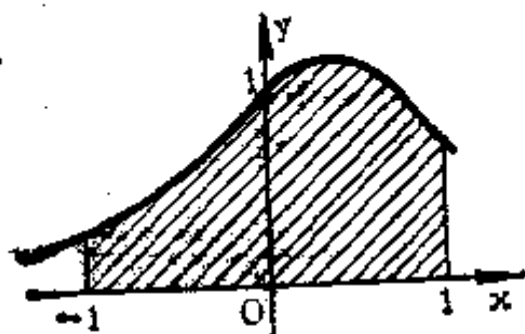


图 4.8

$$\begin{aligned} \text{解} \quad & \int_0^{2\pi} \frac{dx}{1 + e \cos x} \\ &= \int_0^{\pi} \frac{dx}{1 + e \cos x} + \int_{\pi}^{2\pi} \frac{dx}{1 + e \cos x} \\ &= \int_0^{\pi} \frac{dx}{1 + e \cos x} + \int_{\pi}^0 \frac{d(2\pi - x)}{1 + e \cos(2\pi - x)} \\ &= 2 \int_0^{\pi} \frac{dx}{1 + e \cos x} \end{aligned}$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \varepsilon \cos x} + 2 \int_{\frac{\pi}{2}}^{\pi} \frac{dx}{1 + \varepsilon \cos x}$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \varepsilon \cos x} + 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 - \varepsilon \cos x}$$

$$= 4 \int_0^{\frac{\pi}{2}} \frac{dx}{1 - \varepsilon^2 \cos^2 x}$$

$$= 4 \int_0^{\frac{\pi}{2}} \frac{dx}{(1 - \varepsilon^2) \cos^2 x + \sin^2 x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d \operatorname{tg} x}{\operatorname{tg}^2 x + (1 - \varepsilon^2)}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{4}{\sqrt{1 - \varepsilon^2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} x}{\sqrt{1 - \varepsilon^2}} \right)$$

$$= \frac{4}{\sqrt{1 - \varepsilon^2}} \cdot \frac{\pi}{2} = \frac{2\pi}{\sqrt{1 - \varepsilon^2}}$$

2214. $\int_{-1}^1 \frac{dx}{\sqrt{(1 - 2ax + a^2)(1 - 2bx + b^2)}} \quad (|a| < 1,$

$|b| < 1, ab > 0).$

解 在公式

$$\int \frac{dx}{\sqrt{Ax^2 + Bx + C}}$$

$$= \frac{1}{\sqrt{A}} \ln \left| Ax + \frac{B}{2} + \sqrt{A} \cdot \sqrt{Ax^2 + Bx + C} \right| + C^*$$

中, 设

$$Ax^2 + Bx + C = (1 - 2ax + a^2)(1 - 2bx^2 + b^2),$$

两端求导数得

$$Ax + \frac{B}{2} = -b(1 - 2ax + a^2) - a(1 - 2bx + b^2).$$

由此推得, 当 $x=1$ 时, 在对数符号下的表达式的值为

$$\begin{aligned} & -a(1-b)^2 - b(1-a)^2 + 2\sqrt{ab}(1-a)(1-b) \\ & = -(\sqrt{a} - \sqrt{b})^2(1 + \sqrt{ab})^2, \end{aligned}$$

而当 $x=-1$ 时, 得到值 $-(\sqrt{a} - \sqrt{b})^2(1 - \sqrt{ab})^2$. 于是,

$$\begin{aligned} & \int_{-1}^1 \frac{dx}{\sqrt{(1-2ax+a^2)(1-2bx+b^2)}} \\ & = \frac{1}{\sqrt{ab}} \ln \frac{1 + \sqrt{ab}}{1 - \sqrt{ab}}. \end{aligned}$$

*) 利用1850题的结果.

$$2215. \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad (ab \neq 0).$$

$$\begin{aligned} \text{解} \quad & \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \\ & = \frac{1}{|ab|} \operatorname{arc} \operatorname{tg} \left(\frac{|a| \operatorname{tg} x}{|b|} \right) \Big|_0^{\frac{\pi}{2} *} = \frac{\pi}{2|ab|}. \end{aligned}$$

*) 利用2030题的结果.

2216. 设

$$(a) \int_{-1}^1 \frac{dx}{x^2}; \quad (b) \int_0^{2x} \frac{\sec^2 x dx}{2 + \operatorname{tg}^2 x},$$

$$(B) \int_{-1}^1 \frac{d}{dx} \left(\operatorname{arc} \operatorname{tg} \frac{1}{x} \right) dx.$$

说明为甚么运用牛顿—莱布尼兹公式会得到不正确的结果。

解 (a) 若应用公式得

$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_{-1}^1 = -2 < 0.$$

这是不正确的。事实上，由于函数 $f(x) = \frac{1}{x^2} > 0$ ，

所以当积分存在时，其值必大于零。原因在于该函数在区间 $[-1, 1]$ 上有第二类间断点 $x=0$ 。因而不能运用公式。

(b) 若应用公式得

$$\begin{aligned} & \int_0^{2\pi} \frac{\sec^2 x dx}{2 + \operatorname{tg}^2 x} \\ &= \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \left(\frac{\operatorname{tg} x}{\sqrt{2}} \right) \Big|_0^{2\pi} = 0. \end{aligned}$$

但 $\frac{\sec^2 x}{2 + \operatorname{tg}^2 x} > 0$ ，故积分若存在，必为正。原因在于

原函数在 $[0, 2\pi]$ 上 $x = \frac{\pi}{2}$ ， $x = \frac{3\pi}{2}$ 为第一类不连续点，故不能直接运用公式。

(B) 若应用公式得

$$\begin{aligned} & \int_{-1}^1 \frac{d}{dx} \left(\operatorname{arc} \operatorname{tg} \frac{1}{x} \right) dx \\ &= \operatorname{arc} \operatorname{tg} \frac{1}{x} \Big|_{-1}^1 = \frac{\pi}{2} > 0. \end{aligned}$$

这是不正确的，因为 $\frac{d}{dx}(\operatorname{arctg} \frac{1}{x}) = -\frac{1}{1+x^2} < 0$ 。

所以，积分值必为负。原因在于原函数 $\operatorname{arctg} \frac{1}{x}$ 在 $x=0$ 为第一类不连续点，故不能直接运用公式。

2217. 求 $\int_{-1}^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx$ 。

解 我们有

$$\begin{aligned} & \int_{-1}^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx \\ &= \int_{-1}^0 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx + \int_0^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx \\ &= \frac{1}{1+2^{\frac{1}{x}}} \Big|_{-1}^0 + \frac{1}{1+2^{\frac{1}{x}}} \Big|_0^1 = \frac{2}{3}. \end{aligned}$$

注意，被积函数 $\frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right)$ 显然在 $x=0$ 间断，但

易知 $\lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) = 0$ ，故 $x=0$ 是可去间断点。

若我们补充定义被积函数在 $x=0$ 时的值为 0 ，则被积函数在整个 $[-1, 1]$ 上都是连续的，从而积分

$\int_{-1}^1 \frac{d}{dx} \left(\frac{1}{1+2^{\frac{1}{x}}} \right) dx$ 存在。以后，凡是被积函数有

可去间断点的情形，我们都按此法处理，理解为连

续函数的积分。另外，

$$\int_{-1}^0 \frac{d}{dx} \left(\frac{1}{1+2^x} \right) dx = \frac{1}{1+2^x} \Big|_{-1}^0 = \frac{1}{3}$$

应理解为

$$\begin{aligned} \int_{-1}^0 \frac{d}{dx} \left(\frac{1}{1+2^x} \right) dx &= \lim_{\eta \rightarrow 0^+} \int_{-1}^{-\eta} \frac{d}{dx} \left(\frac{1}{1+2^x} \right) dx \\ &= \lim_{\eta \rightarrow 0^+} \frac{1}{1+2^x} \Big|_{-1}^{-\eta} = \frac{1}{3}. \end{aligned}$$

以后，凡是定积分存在而原函数有间断点的情况，都按此理解，省去取极限的式子，但应理解为取极限的结果。

2218. 求 $\int_0^{100\pi} \sqrt{1-\cos 2x} dx$.

$$\begin{aligned} \text{解} \quad & \int_0^{100\pi} \sqrt{1-\cos 2x} dx \\ &= \sum_{k=1}^{100} \sqrt{2} \int_{(k-1)\pi}^{k\pi} \sqrt{\sin^2 x} dx \\ &= \sum_{k=1}^{100} \int_0^{\pi} \sqrt{\sin^2 x} dx \\ &= 100 \sqrt{2} \int_0^{\pi} \sin x dx = 200 \sqrt{2}. \end{aligned}$$

利用定积分求下列和的极限值：

2219. $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n-1}{n^2} \right)$.

解 这是和的极限，该极限即为函数 $f(x)=x$ 在区间

$[0, 1]$ 上的定积分。事实上，函数 $f(x) = x$ 在 $(0, 1)$ 上是连续的，因而可积分。这样便可将 $[0, 1]$ n 等份，并取 ξ_i 为小区间的左端点，这样作出的和的极限就是题中所要求的极限。于是，

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \int_0^1 x dx = \frac{1}{2}. \end{aligned}$$

以下各题不再说明。

$$2220. \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right).$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n+i} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{1+\frac{i}{n}} \cdot \frac{1}{n} \\ &= \int_0^1 \frac{1}{1+x} dx = \ln 2. \end{aligned}$$

$$2221. \lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} \right).$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n}{n^2+i^2} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} \\ &= \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}. \end{aligned}$$

$$2222. \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \cdots + \sin \frac{(n-1)\pi}{n} \right].$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \frac{1}{n} \sin \frac{i\pi}{n} &= \int_0^1 \sin \pi x dx \\ &= -\frac{1}{\pi} \cos \pi x \Big|_0^1 = \frac{2}{\pi}. \end{aligned}$$

$$2223. \quad \lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}} \quad (p > 0).$$

$$\text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^p \cdot \frac{1}{n} = \int_0^1 \frac{1}{x^{p+1}} dx = \frac{1}{p+1}.$$

$$2224. \quad \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{2}{n}} + \dots + \sqrt{1 + \frac{n}{n}} \right).$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \sqrt{1 + \frac{i}{n}} &= \int_0^1 \sqrt{1+x} dx \\ &= \frac{2}{3} (2\sqrt{2} - 1). \end{aligned}$$

$$2225. \quad \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n}.$$

解 由于

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln \frac{\sqrt[n]{n!}}{n} &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\sum_{i=1}^n \ln i \right) - n \ln n \right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \ln \frac{i}{n} \cdot \frac{1}{n} = \int_0^1 \ln x dx^{*}) \\ &= \lim_{x \rightarrow +0} \int_x^1 \ln x dx \\ &= \lim_{x \rightarrow +0} x(\ln x - 1) \Big|_x^1 = -1. \end{aligned}$$

从而,

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = e^{-1} = \frac{1}{e}.$$

*) 参看后面2388题.

$$2226. \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n f \left(a + k \cdot \frac{b-a}{n} \right) \right].$$

$$\begin{aligned} \text{解} \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n f \left(a + k \frac{b-a}{n} \right) \right] \\ = \int_0^1 f \left(a + (b-a)x \right) dx = \frac{1}{b-a} \int_a^b f(x) dx. \end{aligned}$$

弃掉高阶同等无穷小, 求下列和的极限值

$$2227. \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \sin \frac{\pi}{n^2} + \left(1 + \frac{2}{n} \right) \sin \frac{2\pi}{n^2} \right. \\ \left. + \dots + \left(1 + \frac{n-1}{n} \right) \sin \frac{(n-1)\pi}{n^2} \right].$$

解 由于对于一切 $k < n$, $3 < n$ 有

$$\begin{aligned} 0 &\leq \frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2} \leq \operatorname{tg} \frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2} \\ &\leq \operatorname{tg} \frac{k\pi}{n^2} \left(1 - \cos \frac{k\pi}{n^2} \right) \\ &\leq \frac{\sin \frac{k\pi}{n^2}}{\cos \frac{k\pi}{n^2}} \left(1 - \cos \frac{k\pi}{n^2} \right) \\ &\leq \frac{2k\pi}{n^2} \left(1 - \cos \frac{\pi}{n} \right). \end{aligned}$$

从而,

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n-1} \left(1 + \frac{k}{n}\right) \left(\frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2}\right) \\ &\leq \sum_{k=1}^{n-1} \left(1 + \frac{k}{n}\right) \frac{2k\pi}{n^2} \left(1 - \cos \frac{\pi}{n}\right) \\ &\leq 2\pi \left(1 - \cos \frac{\pi}{n}\right) \rightarrow 0 \quad (n \rightarrow +\infty). \end{aligned}$$

于是,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n}\right) \sin \frac{k\pi}{n^2} &= \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n}\right) \frac{k\pi}{n^2} \\ &\rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n}\right) \left(\frac{k\pi}{n^2} - \sin \frac{k\pi}{n^2}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k\pi}{n} + \frac{k^2\pi}{n^2}\right) \\ &= \int_0^1 \pi(x + x^2) dx = \frac{5\pi}{6}. \end{aligned}$$

$$2228. \quad \lim_{n \rightarrow \infty} \sin \frac{\pi}{n} \cdot \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}}.$$

解 由于

$$\sin \frac{\pi}{n} = \frac{\pi}{n} (1 + \alpha_n),$$

式中 $\lim_{n \rightarrow \infty} \alpha_n = 0$.

于是,

$$\lim_{n \rightarrow \infty} \sin \frac{\pi}{n} \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} (1 + \alpha_n) \frac{\pi}{n} \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}} \\
&= \left(\lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{k=1}^n \frac{1}{2 + \cos \frac{k\pi}{n}} \right) \cdot \lim_{n \rightarrow \infty} (1 + \alpha_n) \\
&= \pi \int_0^1 \frac{dx}{2 + \cos \pi x} = \frac{2}{\sqrt{3}} \operatorname{arctg} \left(\frac{\operatorname{tg} \frac{\pi x}{2}}{\sqrt{3}} \right) \Big|_0^1 \\
&= \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

2229. $\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sqrt{(nx+k)(nx+k+1)}}{n^2} \quad (x > 0).$

解 由于

$$\begin{aligned}
0 &\leq \sqrt{\left(x + \frac{k}{n}\right)\left(x + \frac{k+1}{n}\right)} - \left(x + \frac{k}{n}\right) \\
&= \frac{\left(x + \frac{k}{n}\right)\left(x + \frac{k+1}{n}\right) - \left(x + \frac{k}{n}\right)^2}{\sqrt{\left(x + \frac{k}{n}\right)\left(x + \frac{k+1}{n}\right)} + \left(x + \frac{k}{n}\right)} \\
&\leq \frac{1}{2x} \left(x + \frac{k}{n}\right) \cdot \frac{1}{n}.
\end{aligned}$$

故

$$\begin{aligned}
0 &\leq \frac{\sum_{k=1}^n \sqrt{(nx+k)(nx+k+1)}}{n^2} \\
&\quad - \sum_{k=1}^n \frac{1}{n} \left(x + \frac{k}{n}\right) \leq \frac{1}{2xn^2} \sum_{k=1}^n \left(x + \frac{k}{n}\right)
\end{aligned}$$

$$= \frac{1}{2n} + \frac{1}{4x} \left(1 + \frac{1}{n}\right) \frac{1}{n} \rightarrow 0 \quad (n \rightarrow \infty).$$

于是,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sqrt{(nx+k)(nx+k+1)}}{n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(x + \frac{k}{n}\right) = \int_0^1 (x+t) dt = x + \frac{1}{2}. \end{aligned}$$

$$2230. \quad \lim_{n \rightarrow \infty} \left(\frac{2^{\frac{1}{n}}}{n+1} + \frac{2^{\frac{2}{n}}}{n+\frac{1}{2}} + \cdots + \frac{2^{\frac{n}{n}}}{n+\frac{1}{n}} \right).$$

解 由于

$$0 < \frac{1}{n} - \frac{1}{n+\frac{1}{k}} = \frac{1}{n(nk+1)} < \frac{1}{n^2},$$

故

$$\begin{aligned} 0 < \frac{1}{n} \sum_{k=1}^n 2^{\frac{k}{n}} - \sum_{k=1}^n \frac{2^{\frac{k}{n}}}{n+\frac{1}{k}} &< \frac{1}{n^2} \sum_{k=1}^n 2^{\frac{k}{n}} < \frac{2}{n} \rightarrow 0 \\ & (n \rightarrow \infty). \end{aligned}$$

于是,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{n+\frac{1}{k}} \cdot 2^{\frac{k}{n}} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n 2^{\frac{k}{n}} \\ &= \int_0^1 2^x dx = \frac{1}{\ln 2}. \end{aligned}$$

2231. 求: $\frac{d}{dx} \int_a^b \sin x^2 dx$, $\frac{d}{da} \int_a^b \sin x^2 dx$,

$$\frac{d}{db} \int_a^b \sin x^2 dx.$$

解 $\frac{d}{dx} \int_a^b \sin x^2 dx = 0$

$$\begin{aligned} \frac{d}{da} \int_a^b \sin x^2 dx &= -\frac{d}{da} \int_a^a \sin x^2 dx \\ &= -\sin a^2 \end{aligned}$$

$$\frac{d}{db} \int_a^b \sin x^2 dx = \sin b^2.$$

2232. 求: (a) $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t^2} dt$;

(b) $\frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}}$;

(B) $\frac{d}{dx} \int_{\sin x}^{\cos x} \cos(\pi t^2) dt$.

解 (a) $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t^2} dt$

$$= \left(\frac{d}{d(x^2)} \int_0^{x^2} \sqrt{1+t^2} dt \right) \cdot \frac{d}{dx} (x^2)$$

$$= 2x \cdot \sqrt{1+x^4};$$

(b) $\frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}}$

$$\begin{aligned}
&= \frac{d}{dx} \int_{x^2}^0 \frac{dt}{\sqrt{1+t^4}} + \frac{d}{dx} \int_0^{x^3} \frac{dt}{\sqrt{1+t^4}} \\
&= \frac{d}{dx}(x^3) \cdot \frac{d}{d(x^3)} \int_0^{x^3} \frac{dt}{\sqrt{1+t^4}} \\
&\quad - \frac{d}{dx}(x^2) \cdot \frac{d}{d(x^2)} \int_0^{x^2} \frac{dt}{\sqrt{1+t^4}} \\
&= \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}
\end{aligned}$$

$$\begin{aligned}
\text{(B)} \quad &\frac{d}{dx} \int_{\sin x}^{\cos x} \cos(\pi t^2) dt \\
&= \frac{d}{dx} \int_{\sin x}^0 \cos(\pi t^2) dt \\
&\quad + \frac{d}{dx} \int_0^{\cos x} \cos(\pi t^2) dt \\
&= -\frac{d(\sin x)}{dx} \cdot \frac{d}{d(\sin x)} \int_0^{\sin x} \cos(\pi t^2) dt \\
&\quad + \frac{d(\cos x)}{dx} \cdot \frac{d}{d(\cos x)} \int_0^{\cos x} \cos(\pi t^2) dt \\
&= -\cos x \cdot \cos(\pi \sin^2 x) \\
&\quad - \sin x \cdot \cos(\pi \cos^2 x) * \\
&= (\sin x - \cos x) \cdot \cos(\pi \sin^2 x) \\
*) \quad &\cos(\pi \cos^2 x) = \cos(\pi - \pi \sin^2 x) \\
&= -\cos(\pi \sin^2 x)
\end{aligned}$$

2233. 求:

$$(a) \lim_{x \rightarrow 0} \frac{\int_0^x \cos x^2 dx}{x}; \quad (b) \lim_{x \rightarrow +\infty} \frac{\int_0^x (\operatorname{arctg} x)^2 dx}{\sqrt{x^2+1}}$$

$$(B) \lim_{x \rightarrow +\infty} \frac{\left(\int_0^x e^{x^2} dx\right)^2}{\int_0^x e^{2x^2} dx}$$

解 (a) $\lim_{x \rightarrow 0} \frac{\int_0^x \cos x^2 dx}{x} = \lim_{x \rightarrow 0} (\cos x^2) = 1;$

$$(b) \lim_{x \rightarrow +\infty} \frac{\int_0^x (\operatorname{arctg} x)^2 dx}{\sqrt{x^2+1}} = \lim_{x \rightarrow +\infty} \frac{(\operatorname{arctg} x)^2}{\frac{x}{\sqrt{1+x^2}}} \\ = \frac{\pi^2}{4};$$

$$(B) \lim_{x \rightarrow +\infty} \frac{\left(\int_0^x e^{x^2} dx\right)^2}{\int_0^x e^{2x^2} dx}$$

$$= \lim_{x \rightarrow +\infty} \frac{2e^{x^2} \cdot \int_0^x e^{x^2} dx}{e^{2x^2}} = \lim_{x \rightarrow +\infty} \frac{2 \int_0^x e^{x^2} dx}{e^{x^2}}$$

$$= \lim_{x \rightarrow +\infty} \frac{2e^{x^2}}{2xe^{x^2}} = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0.$$

2234. 证明

$$\text{当 } x \rightarrow \infty \text{ 时, } \int_0^x e^{x^2} dx \sim \frac{1}{2x} e^{x^2}.$$

证 由于

$$\lim_{x \rightarrow \infty} \frac{\int_0^x e^{x^2} dx}{\frac{1}{2x} e^{x^2}} = \lim_{x \rightarrow \infty} \frac{e^{x^2}}{e^{x^2} \left(1 - \frac{1}{2x^2}\right)} = 1,$$

所以, 当 $x \rightarrow \infty$ 时,

$$\int_0^x e^{x^2} dx \sim \frac{1}{2x} e^{x^2}.$$

2235. 求:

$$\lim_{x \rightarrow +0} \frac{\int_0^{\sin x} \sqrt{\operatorname{tg} x} dx}{\int_0^{\operatorname{tg} x} \sqrt{\sin x} dx}$$

解
$$\lim_{x \rightarrow +0} \frac{\int_0^{\sin x} \sqrt{\operatorname{tg} x} dx}{\int_0^{\operatorname{tg} x} \sqrt{\sin x} dx}$$

$$= \lim_{x \rightarrow +0} \frac{\sqrt{\operatorname{tg}(\sin x)} (\sin x)'}{\sqrt{\sin(\operatorname{tg} x)} (\operatorname{tg} x)'}$$

$$= \lim_{x \rightarrow +0} \sqrt{\frac{\operatorname{tg}(\sin x)}{\sin x} \cdot \frac{\sin x}{\operatorname{tg} x} \cdot \frac{\operatorname{tg} x}{\sin(\operatorname{tg} x)}}$$

$$\bullet \lim_{x \rightarrow +0} \cos^3 x = 1.$$

2236. 设 $f(x)$ 为连续正值函数, 证明当 $x \geq 0$ 时, 函数

$$\varphi(x) = \frac{\int_0^x t f(t) dt}{\int_0^x f(t) dt}$$

增加.

证 首先注意, $\lim_{x \rightarrow 0^+} \varphi(x) = \lim_{x \rightarrow 0^+} \frac{x f(x)}{f(x)} = 0$, 故若

规定 $\varphi(0) = 0$, 则 $\varphi(x)$ 是 $x \geq 0$ 上的连续函数.

另外,

$$\begin{aligned}\varphi'(x) &= \frac{1}{\left(\int_0^x f(t) dt\right)^2} \left\{ x f(x) \int_0^x f(t) dt \right. \\ &\quad \left. - f(x) \int_0^x t f(t) dt \right\} \\ &= \frac{f(x)}{\left(\int_0^x f(t) dt\right)^2} \cdot \int_0^x (x-t) f(t) dt\end{aligned}$$

> 0 (当 $x > 0$ 时),

所以, $\varphi(x)$ 当 $x \geq 0$ 时是增加的.

2237. 求

$$(a) \int_0^2 f(x) dx, \text{ 设 } f(x) = \begin{cases} x^2, & \text{当 } 0 \leq x \leq 1, \\ 2-x, & \text{当 } 1 < x \leq 2; \end{cases}$$

$$(b) \int_0^1 f(x) dx, \text{ 设 } f(x) = \begin{cases} x, & \text{当 } 0 \leq x \leq t, \\ t \cdot \frac{1-x}{1-t}, & \text{当 } t \leq x \leq 1. \end{cases}$$

$$\begin{aligned}\text{解 (a)} \int_0^2 f(x) dx &= \int_0^1 x^2 dx + \int_1^2 (2-x) dx \\ &= \frac{5}{6}.\end{aligned}$$

$$\begin{aligned}
 (6) \quad \int_0^1 f(x) dx &= \int_0^t x dx + \int_t^1 t \cdot \frac{1-x}{1-t} dx \\
 &= \frac{t}{2}.
 \end{aligned}$$

2238. 计算下列积分并把它们当作参数 α 的函数作出积分 $I = I(\alpha)$ 的图形, 设:

$$(a) \quad I = \int_0^1 x|x-a| dx;$$

$$(b) \quad I = \int_0^\pi \frac{\sin^2 x}{1+2\alpha \cos x + \alpha^2} dx;$$

$$(c) \quad I = \int_0^\pi \frac{\sin x dx}{\sqrt{1+2\alpha \cos x + \alpha^2}}.$$

解 (a) 分三种情况:

1° 若 $\alpha < 0$, 则

$$I = \int_0^1 x(x-\alpha) dx = \frac{1}{3} - \frac{\alpha}{2};$$

2° 若 $\alpha > 1$, 则

$$I = \int_0^1 x(\alpha-x) dx = \frac{\alpha}{2} - \frac{1}{3};$$

3° 若 $0 \leq \alpha \leq 1$, 则

$$\begin{aligned}
 I &= \int_0^\alpha x(\alpha-x) dx + \int_\alpha^1 x(x-\alpha) dx \\
 &= \frac{\alpha^3}{3} - \frac{\alpha}{2} + \frac{1}{3}.
 \end{aligned}$$

于是,

$$\int_0^1 x|x-a| dx = \begin{cases} \frac{1}{3} - \frac{a}{2}, & \text{当 } a < 0, \\ \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3}, & \text{当 } 0 \leq a \leq 1, \\ \frac{a}{2} - \frac{1}{3}, & \text{当 } a > 1 \text{ (图4.9)}. \end{cases}$$

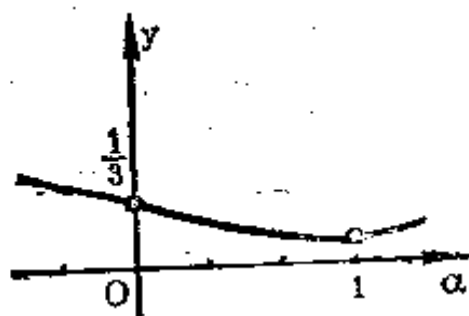


图 4.9

(6) 分两种情况:

1° 若 $|a| \leq 1$, 则

$$\begin{aligned} I &= \int_0^x \frac{\sin^2 x}{1+2\cos x+a^2} dx \\ &= \frac{1}{4a^2} \int_0^x \frac{4a^2(1-\cos^2 x) dx}{(1+a^2)+2a\cos x} \\ &= \frac{1}{4a^2} \int_0^x \frac{[(1+a^2)^2-4a^2\cos^2 x]+[4a^2-(1+a^2)^2]}{(1+a^2)+2a\cos x} dx \\ &= \frac{1}{4a^2} \int_0^x [(1+a^2)-2a\cos x] dx - \frac{(1-a^2)^2}{4a^2} \\ &\quad \cdot \int_0^x \frac{dx}{(1+a^2)+2a\cos x} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\alpha^2} \left[(1+\alpha^2)x - 2\alpha \sin x \right] \Big|_0^\pi - \frac{(1-\alpha^2)^2}{4\alpha^2} \\
&\quad \cdot \frac{2}{1-\alpha^2} \cdot \operatorname{arc\,tg} \left(\sqrt{\frac{1+\alpha^2-2\alpha}{1+\alpha^2+2\alpha}} \operatorname{tg} \frac{x}{2} \right) \Big|_0^\pi \\
&= \frac{(1+\alpha^2)\pi}{4\alpha^2} - \frac{(1-\alpha^2)\pi}{4\alpha^2} = \frac{\pi}{2}.
\end{aligned}$$

2° 若 $|\alpha| > 1$, 则同上述情况类似有

$$\begin{aligned}
I &= \frac{(1+\alpha^2)\pi}{4\alpha^2} - \frac{(\alpha^2-1)^2}{4\alpha^2} \\
&\quad \cdot \frac{2}{\alpha^2-1} \operatorname{arc\,tg} \left(\sqrt{\frac{1+\alpha^2-2\alpha}{1+\alpha^2+2\alpha}} \operatorname{tg} \frac{x}{2} \right) \Big|_0^\pi \\
&= \frac{(1+\alpha^2)\pi}{4\alpha^2} - \frac{(\alpha^2-1)\pi}{4\alpha^2} = \frac{\pi}{2\alpha^2}.
\end{aligned}$$

于是,

$$\begin{aligned}
&\int_0^\pi \frac{\sin^2 x dx}{1+2\alpha \cos x + \alpha^2} \\
&= \begin{cases} \frac{\pi}{2}, & \text{当 } |\alpha| \leq 1; \\ \frac{\pi}{2\alpha^2}, & \text{当 } |\alpha| > 1, \end{cases} \quad (\text{图4.10})
\end{aligned}$$

* 利用2028题(a)的结果.

$$\begin{aligned}
(\text{B}) \quad &\int_0^\pi \frac{\sin x dx}{\sqrt{1-2\alpha \cos x + \alpha^2}} \\
&= \frac{1}{\alpha} \sqrt{1+\alpha^2-2\alpha \cos x} \Big|_0^\pi
\end{aligned}$$

$$= \begin{cases} 2, & \text{当 } |a| \leq 1, \\ \frac{2}{|a|}, & \text{当 } |a| > 1. \end{cases} \quad (\text{图 4.11})$$

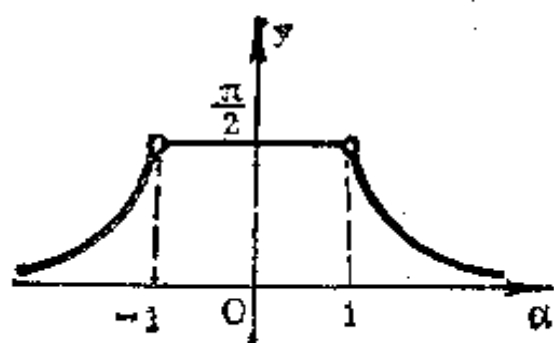


图 4.10

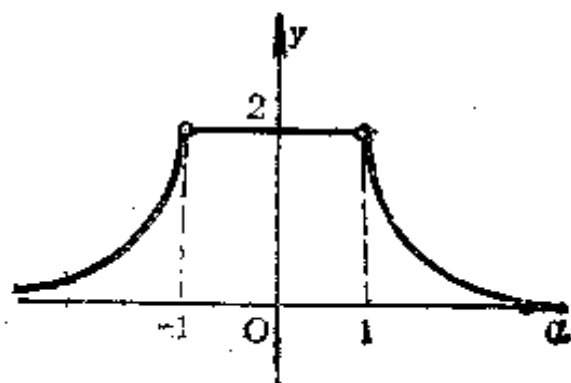


图 4.11

利用部分积分法的公式，求下列定积分：

2239. $\int_0^{\ln 2} x e^{-x} dx.$

$$\begin{aligned} \text{解} \quad \int_0^{\ln 2} x e^{-x} dx &= - \int_0^{\ln 2} x d(e^{-x}) \\ &= -x e^{-x} \Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x} dx \\ &= -\frac{1}{2} \ln 2 - e^{-x} \Big|_0^{\ln 2} \\ &= -\frac{1}{2} \ln 2 + \frac{1}{2} = \frac{1}{2} \ln \frac{e}{2}. \end{aligned}$$

2240. $\int_0^{\pi} x \sin x dx.$

$$\text{解} \quad \int_0^{\pi} x \sin x dx = -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx = \pi.$$

$$2241. \int_0^{2\pi} x^2 \cos x dx.$$

$$\begin{aligned} \text{解 } \int_0^{2\pi} x^2 \cos x dx &= x^2 \sin x \Big|_0^{2\pi} - 2 \int_0^{2\pi} x \sin x dx \\ &= 2 \left(x \cos x \Big|_0^{2\pi} - \int_0^{2\pi} \cos x dx \right) \\ &= 4\pi. \end{aligned}$$

$$2242^+. \int_{\frac{1}{e}}^e |\lg x| dx.$$

$$\begin{aligned} \text{解 } \int_{\frac{1}{e}}^e |\lg x| dx &= \int_{\frac{1}{e}}^1 (-\lg x) dx + \int_1^e \lg x dx \\ &= \left(-x \lg x \Big|_{\frac{1}{e}}^1 + \int_{\frac{1}{e}}^1 \frac{1}{\ln 10} dx \right) + x \lg x \Big|_1^e \\ &\quad - \int_1^e \frac{1}{\ln 10} dx \\ &= 2 \left(1 - \frac{1}{e} \right) \lg e. \end{aligned}$$

$$2243. \int_0^1 \arccos x dx.$$

$$\begin{aligned} \text{解 } \int_0^1 \arccos x dx &= x \arccos x \Big|_0^1 - \lim_{\varepsilon \rightarrow 0} \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} dx \end{aligned}$$

$$= -\lim_{x \rightarrow 0} \sqrt{1-x^2} \Big|_0^{1-\epsilon} = 1.$$

2244⁺. $\int_0^{\sqrt{3}} x \operatorname{arc} \operatorname{tg} x dx.$

解 $\int_0^{\sqrt{3}} x \operatorname{arc} \operatorname{tg} x dx$

$$= \frac{1}{2} x^2 \operatorname{arc} \operatorname{tg} x \Big|_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1+x^2} dx$$

$$= \frac{3}{2} \operatorname{arc} \operatorname{tg} \sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2} \operatorname{arc} \operatorname{tg} \sqrt{3}$$

$$= 2 \operatorname{arc} \operatorname{tg} \sqrt{3} - \frac{\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

利用适当的变数代换，求下列定积分：

2245. $\int_{-1}^1 \frac{x dx}{\sqrt{5-4x}}.$

解 设 $\sqrt{5-4x} = t$ ，则

$$\int_{-1}^1 \frac{x dx}{\sqrt{5-4x}} = \int_3^1 \frac{5-t^2}{8} dt = \frac{1}{6}.$$

2246. $\int_0^a x^2 \sqrt{a^2-x^2} dx \quad (a > 0).$

解 设 $x = a \sin t$ ，则

$$\int_0^a x^2 \sqrt{a^2-x^2} dx = a^4 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt$$

$$= \frac{a^4}{4} \int_0^{\frac{\pi}{2}} \sin^2 2t dt = \frac{a^4}{8} \left(t - \frac{1}{4} \sin 4t \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^4}{16}.$$

$$2247. \int_0^{0.75} \frac{dx}{(x+1)\sqrt{x^2+1}}.$$

解 设 $t = \frac{1}{x+1}$, 则

$$\begin{aligned} & \int_0^{0.75} \frac{dx}{(x+1)\sqrt{x^2+1}} \\ &= \int_{\frac{4}{7}}^1 \frac{dt}{\sqrt{2t^2-2t+1}} \\ &= \frac{1}{\sqrt{2}} \ln (2t-1 + \sqrt{2t^2-2t+2}) \Big|_{\frac{4}{7}}^1 \\ &= \frac{1}{\sqrt{2}} \ln \frac{1 + \sqrt{2}}{\frac{1}{7} + \sqrt{\frac{50}{49}}} = \frac{1}{\sqrt{2}} \ln \frac{7 + 7\sqrt{2}}{1 + 5\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \ln \frac{9 + 4\sqrt{2}}{7}. \end{aligned}$$

$$2248. \int_0^{\ln 2} \sqrt{e^x-1} dx.$$

解 设 $\sqrt{e^x-1} = t$, 则

$$\begin{aligned} & \int_0^{\ln 2} \sqrt{e^x-1} dx \\ &= 2 \int_0^1 \frac{t^2 dt}{1+t^2} = 2 (t - \operatorname{arc} \operatorname{tg} t) \Big|_0^1 = 2 - \frac{\pi}{2}. \end{aligned}$$

$$2249. \int_0^1 \frac{\operatorname{arc} \sin \sqrt{x}}{\sqrt{x(1-x)}} dx.$$

解 设 $\sqrt{x} = t$, 则

$$\int_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} dx$$

$$= 2 \int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} dt = (\arcsin t)^2 \Big|_0^1 = \frac{\pi^2}{4}.$$

2250. 假设 $x - \frac{1}{x} = t$, 来计算积分

$$\int_{-1}^1 \frac{1+x^2}{1+x^4} dx.$$

解 由于被积函数是偶函数, 于是,

$$\int_{-1}^1 \frac{1+x^2}{1+x^4} dx$$

$$= 2 \int_0^1 \frac{1+x^2}{1+x^4} dx = \lim_{N \rightarrow \infty} 2 \int_N^0 \frac{dt}{t^2+2}$$

$$= \lim_{N \rightarrow \infty} \sqrt{2} \arctg \frac{t}{\sqrt{2}} \Big|_N^0 = \frac{\pi}{\sqrt{2}}.$$

2251. 设:

$$(a) \int_{-1}^1 dx, \quad t = x^{\frac{2}{3}};$$

$$(b) \int_{-1}^1 \frac{dx}{1+x^2}, \quad x = \frac{1}{t};$$

$$(B) \int_0^{\pi} \frac{dx}{1+\sin^2 x}, \quad \operatorname{tg} x = t.$$

说明为甚么用 $\varphi(t)$ 代换 x 会引致不正确的结果.

解 (a) $\int_{-1}^1 dx = 2$. 但若作代换 $t = x^{\frac{2}{3}}$, 则得

$$\int_{-1}^1 dx = \pm \frac{3}{2} \int_1^1 t^{\frac{1}{2}} dt = 0.$$

其错误在于代换 $t = x^{\frac{2}{3}}$ 的反函数 $x = \pm t^{\frac{3}{2}}$ 不是单值的。

$$(6) \int_{-1}^1 \frac{dx}{1+x^2} = \operatorname{arc} \operatorname{tg} x \Big|_{-1}^1 = \frac{\pi}{2}. \text{ 但若作代换}$$

$x = \frac{1}{t}$, 则得

$$\int_{-1}^1 \frac{dx}{1+x^2} = - \int_{-1}^1 \frac{dt}{1+t^2},$$

于是得出错误的结果: $\int_{-1}^1 \frac{dx}{1+x^2} = 0.$

其错误在于 $x = \frac{1}{t}$, 当 $t = 0$ (0 属于 $[-1, 1]$) 时不连续。

$$(B) \int_0^{\pi} \frac{dx}{1+\sin^2 x} \text{ 大于零, 但若作代换 } t = \operatorname{tg} x,$$

则得

$$\int_0^{\pi} \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} (\sqrt{2} \operatorname{tg} x) \Big|_0^{\pi} = 0.$$

其错误在于 $t = \operatorname{tg} x$ 在 $x = \frac{\pi}{2}$ 处不连续。

2252. 在积分

$$\int_0^{\frac{3}{2}} x \sqrt{1-x^2} dx$$

中, 令 $x = \sin t$ 是否可以?

解 不可以, 因为 $\sin t = x$ 不可能大于 1.

2253. 于积分 $\int_0^1 \sqrt{1-x^2} dx$ 中, 当作变数的代换 $x = \sin t$

时, 可否取数 π 和 $\frac{\pi}{2}$ 作为新的上下限?

解 可以. 因为满足定积分换元的条件.

事实上,

$$\begin{aligned} \int_0^1 \sqrt{1-x^2} dx &= \int_{\pi}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t} d(\sin t) \\ &= \int_{\pi}^{\frac{\pi}{2}} |\cos t| \cos t dt = - \int_{\pi}^{\frac{\pi}{2}} \cos^2 t dt \\ &= \left(-\frac{t}{2} + \frac{\sin 2t}{4} \right) \Big|_{\pi}^{\frac{\pi}{2}} = \frac{\pi}{4}. \end{aligned}$$

2254. 证明: 若函数 $f(x)$ 于闭区间 (a, b) 内连续, 则

$$\int_a^b f(x) dx = (b-a) \int_0^1 f[a+(b-a)x] dx.$$

证 设 $x = a + (b-a)t$, 则 $dx = (b-a)dt$.

代入得

$$\int_a^b f(x) dx = (b-a) \int_0^1 f[a+(b-a)t] dt,$$

即

$$\int_a^b f(x) dx = (b-a) \int_0^1 f[a+(b-a)x] dx.$$

2255. 证明: 等式

$$\int_0^a x^3 f(x^2) dx = \frac{1}{2} \int_0^{a^2} x f(x) dx \quad (a > 0).$$

证 设 $x = \sqrt{t}$, 则

$$\int_0^a x^3 f(x^2) dx$$

$$= \int_0^{a^2} t^{\frac{3}{2}} f(t) \cdot \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^{a^2} t f(t) dt.$$

即

$$\int_0^a x^3 f(x^2) dx = \frac{1}{2} \int_0^{a^2} x f(x) dx.$$

2256. 设 $f(x)$ 为闭区间 $[A, B] \supset [a, b]$ 上的连续函数, 当

$$A - a < x < B - b \text{ 时, 求 } \frac{d}{dx} \int_a^b f(x+y) dy.$$

$$\text{解 } \frac{d}{dx} \int_a^b f(x+y) dy$$

$$= \frac{d}{dx} \int_{a+x}^{b+x} f(y) dy = f(b+x) - f(a+x).$$

2257. 证明: 若函数 $f(x)$ 于闭区间 $[0, 1]$ 上连续, 则

$$(a) \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx;$$

$$(b) \int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx.$$

证: (a) 设 $\frac{\pi}{2} - t = x$, 则 $dx = -dt$, 且

$$f(\sin x) = f(\cos t).$$

代入得

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_{\frac{\pi}{2}}^0 f(\cos t) dt$$

$$= \int_0^{\frac{\pi}{2}} f(\cos t) dt,$$

即

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx.$$

(6) 设 $\pi - t = x$, 则 $dx = -dt$, 且
 $xf(\sin x) = (\pi - t)f(\sin t)$.

代入得

$$\begin{aligned} \int_0^{\pi} xf(\sin x) dx &= - \int_{\pi}^0 (\pi - t)f(\sin t) dt \\ &= \pi \int_0^{\pi} f(\sin t) dt - \int_0^{\pi} t f(\sin t) dt, \end{aligned}$$

即

$$\int_0^{\pi} xf(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx.$$

2258. 证明: 若函数 $f(x)$ 于闭区间 $[-l, l]$ 上连续, 则

(1) 若函数 $f(x)$ 为偶函数时,

$$\int_{-l}^l f(x) dx = 2 \int_0^l f(x) dx,$$

(2) 若函数 $f(x)$ 为奇函数时,

$$\int_{-l}^l f(x) dx = 0.$$

给出这些事实的几何解释.

证 (1) 由于 $f(x)$ 为偶函数, 即 $f(x) = f(-x)$,
 $(x \in (-l, l))$. 于是设 $x = -t$, 则有

$$\begin{aligned}
 \int_{-1}^1 f(x) dx &= \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx \\
 &= - \int_{-1}^0 f(-x) d(-x) + \int_0^1 f(x) dx \\
 &= - \int_1^0 f(t) dt + \int_0^1 f(x) dx \\
 &= \int_0^1 f(t) dt + \int_0^1 f(x) dx = 2 \int_0^1 f(x) dx.
 \end{aligned}$$

其几何解释如下:

由于 $f(x) = f(-x)$, 故图形关于 Oy 轴对称. 于是由曲线 $y = f(x)$,

直线 $x = -1$ 及 $x = 1$ 所围成图形的面积为

由曲线 $y = f(x)$, 直线 $x = 0$ 及 $x = 1$ 所围成的图形的面积 S

的两倍(图4.12).

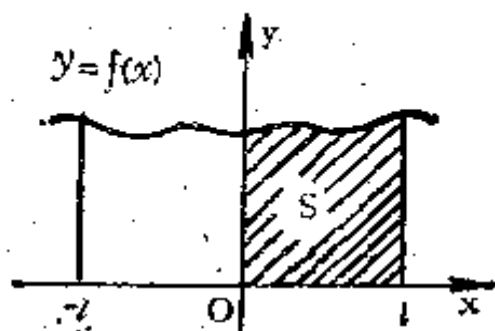


图 4.12

(2) 由于 $f(x) = -f(-x)$, 设 $x = -t$, 则

$$\begin{aligned}
 &\int_{-1}^1 f(x) dx \\
 &= - \int_{-1}^0 f(-x) dx + \int_0^1 f(x) dx \\
 &= \int_1^0 f(t) dt + \int_0^1 f(x) dx = 0.
 \end{aligned}$$

其几何解释如下:

由于 $f(x) = -f(-x)$, 故图形关于原点对称, 于是由 -1 到 0 之间所围之面积, 与由 0 到 1 之间所围成之面积绝对值相等, 符号相反, 故其面积的代数和为零 (图 4.13).

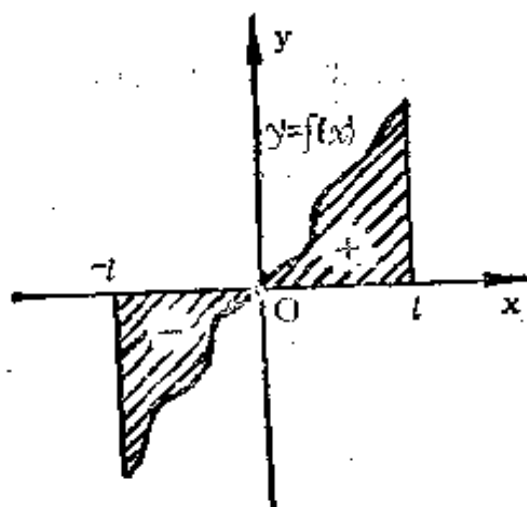


图 4.13

2259. 证明: 偶函数的原函数中之一个为奇函数, 而奇函数的一切原函数皆为偶函数.

证 设 $f(x)$ 在 $(-1, 1)$ 上定义^{*)}, 且 $F(x)$ 是 $f(x)$ 的一个原函数, 当 $f(-x) = f(x)$ 时, 由于

$$f(x) = \frac{d}{dx} F(x) \text{ 及 } f(-x) = -\frac{d}{dx} F(-x),$$

故有 $\frac{d}{dx} [F(x) + F(-x)] = 0$. 从而可得

$$F(x) + F(-x) = C_1, \text{ 且 } C_1 = 2F(0).$$

于是, $f(x)$ 有一个原函数 $F(x) - F(0)$ 是奇函数.

当 $f(-x) = -f(x)$ 时, 类似地可得

$$F(x) - F(-x) = C_2, \text{ 且 } C_2 = 0.$$

于是, $F(-x) = F(x)$, 即 $f(x)$ 的任一原函数 $F(x) + C$ (C 为任意常数) 也为偶函数.

^{*)} 如果 $f(x)$ 在 $(-1, 1)$ 上可积, 则由

$$F_c(x) = \int_0^x f(t) dt + C \quad (C \text{ 是任意常数})$$

也可获证，其中 $F_2(x)$ 为 $f(x)$ 的全部原函数。

2260. 引入新变数

$$t = x + \frac{1}{x}.$$

来计算积分 $\int_{\frac{1}{2}}^2 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx$ 。

解 设 $t = x + \frac{1}{x}$ ，则

$$t^2 - 4 = \left(x - \frac{1}{x}\right)^2, \quad x = \frac{1}{2}(t \pm \sqrt{t^2 - 4}).$$

于是，

$$\begin{aligned} & \int_{\frac{1}{2}}^2 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx \\ &= \int_1^2 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx \\ & \quad + \int_{\frac{1}{2}}^1 \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx \\ &= \int_2^{\frac{5}{2}} (1 + \sqrt{t^2 - 4}) e^t d\left[\frac{1}{2}(t + \sqrt{t^2 - 4})\right] \\ & \quad + \int_{\frac{5}{2}}^2 (1 - \sqrt{t^2 - 4}) e^t d\left[\frac{1}{2}(t - \sqrt{t^2 - 4})\right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_2^{\frac{5}{2}} (1 + \sqrt{t^2 - 4}) e^t \left(1 + \frac{t}{\sqrt{t^2 - 4}}\right) dt \\
&\quad - \frac{1}{2} \int_2^{\frac{5}{2}} (1 - \sqrt{t^2 - 4}) e^t \left(1 - \frac{t}{\sqrt{t^2 - 4}}\right) dt \\
&= \int_2^{\frac{5}{2}} e^t \left[\sqrt{t^2 - 4} + \frac{t}{\sqrt{t^2 - 4}} \right] dt \\
&= \int_2^{\frac{5}{2}} (\sqrt{t^2 - 4} d(e^t) + e^t d\sqrt{t^2 - 4}) \\
&= (\sqrt{t^2 - 4}) e^t \Big|_2^{\frac{5}{2}} = \frac{3}{2} e^{\frac{5}{2}}.
\end{aligned}$$

2261. 于积分

$$\int_0^{2\pi} f(x) \cos x dx.$$

中实行变数代换 $\sin x = t$.

$$\begin{aligned}
\text{解} \quad &\int_0^{2\pi} f(x) \cos x dx \\
&= \int_0^{\frac{\pi}{2}} f(x) \cos x dx + \int_{\frac{\pi}{2}}^{\pi} f(x) \cos x dx \\
&\quad + \int_{\pi}^{\frac{3\pi}{2}} f(x) \cos x dx + \int_{\frac{3\pi}{2}}^{2\pi} f(x) \cos x dx.
\end{aligned}$$

在右端的第一个积分中, 设 $\sin x = t$; 第二、第三个积分中, 设 $\sin(\pi - x) = t$; 第四个积分中, 设 $\sin(2\pi - x) = -t$, 代入得

$$\int_0^{2\pi} f(x) \cos x dx$$

$$= \int_0^1 [f(\arcsin t) - f(\pi - \arcsin t)] dt \\ + \int_{-1}^0 [f(2\pi + \arcsin t) - f(\pi - \arcsin t)] dt.$$

2262. 计算积分

$$\int_{e^{-2\pi n}}^1 \left| \left[\cos \left(\ln \frac{1}{x} \right) \right]' \right| dx,$$

式中 n 为自然数.

证 $\left[\cos \left(\ln \frac{1}{x} \right) \right]' = \frac{\sin(-\ln x)}{x}$. 设 $x = e^{-t}$,

则 $dx = -e^{-t} dt$, $\frac{\sin(-\ln x)}{x} = \frac{\sin t}{e^{-t}} = e^t \sin t$.

代入得

$$\int_{e^{-2\pi n}}^1 \left| \left[\cos \left(\ln \frac{1}{x} \right) \right]' \right| dx = \int_0^{2\pi n} |\sin t| dt \\ = \sum_{k=1}^{2n} \int_{(k-1)\pi}^{k\pi} |\sin t| dt = \sum_{k=1}^{2n} \int_0^{\pi} \sin t dt \\ = 2 \cdot 2n = 4n.$$

2263. 求:

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$$

解 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$ *) \\ $= \frac{\pi}{2} \left[-\arctg(\cos x) \right] \Big|_0^{\pi} = \frac{\pi^2}{4}.$

*) 利用2257题结果。

2264. 设

$$f(x) = \frac{(x+1)^2(x-1)}{x^3(x-2)}.$$

求 $\int_{-1}^3 \frac{f'(x)}{1+f^2(x)} dx.$

解 $\int_{-1}^3 \frac{f'(x)}{1+f^2(x)} dx$

$$= \int_{-1}^0 \frac{f'(x)}{1+f^2(x)} dx + \int_0^2 \frac{f'(x)}{1+f^2(x)} dx$$
$$+ \int_2^3 \frac{f'(x)}{1+f^2(x)} dx$$
$$= \operatorname{arc} \operatorname{tg} f(x) \Big|_{-1}^0 + \operatorname{arc} \operatorname{tg} f(x) \Big|_0^2 + \operatorname{arc} \operatorname{tg} f(x) \Big|_2^3$$
$$= \left(-\frac{\pi}{2} - 0\right)^{*)} + \left(-\frac{\pi}{2} - \frac{\pi}{2}\right)$$
$$+ \left(\operatorname{arc} \operatorname{tg} \frac{4^2 \cdot 2}{3^3 \cdot 1} - \frac{\pi}{2}\right)$$
$$= \operatorname{arc} \operatorname{tg} \frac{32}{27} - 2\pi.$$

*) 参看2217题后的注意。

2265. 证明: 若 $f(x)$ 为定义在 $-\infty < x < +\infty$ 而周期为 T 的连续的周期函数, 则

$$\int_a^{a+T} f(x) dx = \int_0^T f(x) dx.$$

式中 a 为任意的数。

$$\begin{aligned} \text{证 } & \int_a^{a+T} f(x) dx \\ &= \int_a^0 f(x) dx + \int_0^T f(x) dx + \int_T^{a+T} f(x) dx. \end{aligned}$$

对上述等式右端的第三个积分, 设 $x-T=t$, 则

$$\int_T^{a+T} f(x) dx = \int_0^a f(t+T) dt = \int_0^a f(t) dt.$$

于是,

$$\int_a^{a+T} f(x) dx = \int_0^T f(x) dx.$$

2266. 证明: 当 n 为奇数时, 函数

$$F(x) = \int_0^x \sin^n x dx \text{ 及 } G(x) = \int_0^x \cos^n x dx$$

为以 2π 为周期的周期函数; 而当 n 为偶数时, 则其中的任何一个皆为线性函数与周期函数的和.

证 当 n 为奇数时, $\sin^n x$ 是奇函数, 而且是以 2π 为周期的函数. 于是,

$$\begin{aligned} F(x+2\pi) &= \int_0^{x+2\pi} \sin^n x dx \\ &= \int_0^{2\pi} \sin^n x dx + \int_{2\pi}^{2\pi+x} \sin^n x dx \\ &= \int_{-\pi}^{\pi} \sin^n(\pi-x) dx + \int_0^x \sin^n x dx \\ &= 0 + \int_0^x \sin^n x dx = F(x) \end{aligned}$$

及

$$\begin{aligned}G(x+2\pi) &= G(x) + \int_0^{2\pi} \cos^n x dx \\&= G(x) + \int_0^\pi \cos^n x dx + \int_\pi^{2\pi} \cos^n x dx \\&= G(x) + \int_0^\pi \cos^n x dx + \int_0^\pi \cos^n(x+\pi) dx \\&= G(x),\end{aligned}$$

从而得知： $F(x)$ 和 $G(x)$ 都是以 2π 为周期的周期函数。
当 n 为偶数时，显然有

$$\begin{aligned}F(x+2\pi) &= F(x) + \int_0^{2\pi} \sin^n x dx, \\G(x+2\pi) &= G(x) + \int_0^{2\pi} \cos^n x dx,\end{aligned}$$

但因

$$\int_0^{2\pi} \sin^n x dx = \int_0^{2\pi} \cos^n x dx = a > 0,$$

所以， $F(x)$ 、 $G(x)$ 都不是 2π 为周期的周期函数。
设

$$F_1(x) = F(x) - \frac{a}{2\pi}x,$$

则

$$\begin{aligned}F_1(x+2\pi) &= F(x+2\pi) - \frac{a}{2\pi}(x+2\pi) \\&= F(x) + a - \frac{a}{2\pi}x - a.\end{aligned}$$

$$= F(x) - \frac{\alpha}{2\pi}x = F_1(x).$$

即 $F_1(x)$ 是以 2π 为周期的函数，而

$$F(x) = F_1(x) + \frac{\alpha}{2\pi}x.$$

所以， $F(x)$ 为周期函数与线性函数之和。

同理，可以证明 $G(x)$ 也是周期函数与线性函数之和。

2267. 证明：函数

$$F(x) = \int_{x_0}^x f(x) dx$$

（式中 $f(x)$ 为具周期 T 的连续的周期函数）在一般的情形下是线性函数与周期函数之和。

证 因为 $F(x) = \int_{x_0}^x f(x) dx$ ，所以

$$F(x+T) - F(x) = \int_x^{x+T} f(x) dx.$$

又因 $f(x)$ 是一周期为 T 的连续函数，所以

$$\int_x^{x+T} f(x) dx = \int_{x_0}^{x_0+T} f(x) dx = K.$$

于是， $F(x+T) - F(x) = K$ 。

如果 $K = 0$ ，则 $F(x)$ 为一周期函数。

如果 $K \neq 0$ ，可考虑函数 $\varphi(x) = F(x) - \frac{K}{T}x$ ，

则因

$$\varphi(x+T) = F(x+T) - \frac{K}{T}(x+T)$$

$$= F(x+T) - \frac{K}{T}x - K$$

$$= F(x) - \frac{K}{T}x = \varphi(x),$$

所以, $\varphi(x)$ 也为一周期函数, 从而

$$F(x) = \varphi(x) + \frac{K}{T}x,$$

即 $F(x)$ 是线性函数与周期等于 T 的周期函数之和.

计算下列积分:

$$2268. \int_0^1 x(2-x^2)^{12} dx.$$

$$\text{解} \int_0^1 x(2-x^2)^{12} dx$$

$$= -\frac{1}{26} (2-x^2)^{13} \Big|_0^1 = 315 \frac{1}{26}.$$

$$2269. \int_{-1}^1 \frac{x dx}{x^2+x+1}.$$

$$\text{解} \int_{-1}^1 \frac{x dx}{x^2+x+1}$$

$$= \frac{1}{2} \int_{-1}^1 \frac{2x+1}{x^2+x+1} dx - \frac{1}{2} \int_{-1}^1 \frac{dx}{\frac{3}{4} + \left(x + \frac{1}{2}\right)^2}$$

$$= \frac{1}{2} \ln(x^2+x+1) \Big|_{-1}^1 - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \Big|_{-1}^1$$

$$= \frac{1}{2} \ln 3 - \frac{\pi}{2\sqrt{3}}.$$

$$2270^+ \cdot \int_1^e (x \ln x)^2 dx.$$

解 $\int_1^e (x \ln x)^2 dx$

$$= x^3 \ln^2 x \Big|_1^e - 2 \int_1^e x^2 \ln x \cdot (1 + \ln x) dx$$

$$= e^3 - 2 \int_1^e x^2 \ln x dx - 2 \int_1^e (x \ln x)^2 dx.$$

移项合并得

$$\int_1^e (x \ln x)^2 dx = \frac{e^3}{3} - \frac{2}{3} \int_1^e x^2 \ln x dx$$

$$= \frac{e^3}{3} - \left(\frac{2}{9} x^3 \ln x - \frac{2}{27} x^3 \right) \Big|_1^e$$

$$= \frac{5}{27} e^3 - \frac{2}{27}.$$

$$2271. \int_1^9 x \sqrt[3]{1-x} dx.$$

解 设 $\sqrt[3]{1-x} = t$, 则

$$\int_1^9 x \sqrt[3]{1-x} dx$$

$$= -3 \int_0^{-2} (t^3 - t^6) dt = -66 \frac{6}{7}.$$

$$2272^+ \cdot \int_{-2}^{-1} \frac{dx}{x \sqrt{x^2 - 1}}.$$

解 设 $x = \frac{1}{t}$, 则

$$\int_{-2}^{-1} \frac{dx}{x\sqrt{x^2-1}}$$

$$= \int_{-\frac{1}{2}}^{-1} \frac{dt}{\sqrt{1-t^2}} = \arcsin t \Big|_{-\frac{1}{2}}^{-1} = -\frac{\pi}{3}.$$

2273. $\int_0^1 x^{15} \sqrt{1+3x^8} dx.$

解 设 $1+3x^8=t$, 则 $24x^7 dx=dt$, $x^8=\frac{1}{3}(t-1)$.

于是,

$$\int_0^1 x^{15} \sqrt{1+3x^8} dx$$

$$= \frac{1}{72} \int_1^4 (t-1) t^{\frac{1}{2}} dt = \frac{29}{270}.$$

2274. $\int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx.$

解 $\int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx$

$$= x \arcsin \sqrt{\frac{x}{1+x}} \Big|_0^3 - \int_0^3 \frac{\sqrt{x} dx}{2(1+x)}$$

$$= \pi - \int_0^{\sqrt{3}} \frac{t^2 dt}{1+t^2} \quad *) = \pi - (t - \arctan t) \Big|_0^{\sqrt{3}}$$

$$= \frac{4\pi}{3} - \sqrt{3}.$$

*) 设 $\sqrt{x}=t$.

2275. $\int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)}.$

$$\begin{aligned}
& \text{解} \int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)} \\
&= \int_0^{2\pi} \frac{dx}{2+\cos x} - \int_0^{2\pi} \frac{dx}{3+\cos x} \\
&= \int_0^{\pi} \frac{dx}{2+\cos x} + \int_0^{\pi} \frac{dx}{2-\cos x} \\
&\quad - \int_0^{2\pi} \frac{dx}{3+\cos x} \\
&= 4 \int_0^{\pi} \frac{dx}{4-\cos^2 x} - 6 \int_0^{\pi} \frac{dx}{9-\cos^2 x} \\
&= 8 \int_0^{\frac{\pi}{2}} \frac{dx}{4\sin^2 x + 3\cos^2 x} \\
&\quad - 12 \int_0^{\frac{\pi}{2}} \frac{dx}{9\sin^2 x + 8\cos^2 x} \\
&= 8 \frac{1}{2\sqrt{3}} \operatorname{arc\,tg} \frac{2\operatorname{tg} x}{\sqrt{3}} \Big|_0^{\frac{\pi}{2}} \\
&\quad - 12 \frac{1}{3\sqrt{8}} \operatorname{arc\,tg} \frac{3\operatorname{tg} x}{\sqrt{8}} \Big|_0^{\frac{\pi}{2}} \\
&= \pi \left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{2}} \right).
\end{aligned}$$

$$2276. \int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x}.$$

$$\begin{aligned}
& \text{解} \int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x} = 8 \int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x} \\
&= \frac{1}{\sqrt{2}} \operatorname{arc\,tg} \left(\frac{\operatorname{tg} 2x}{\sqrt{2}} \right) \Big|_0^{2\pi} = 2\pi\sqrt{2}.
\end{aligned}$$

*) 利用2035题的结果.

$$2277. \int_0^{\frac{\pi}{2}} \sin x \sin 2x \sin 3x dx.$$

解 $\sin x \sin 2x \sin 3x$

$$= \frac{1}{2} (\cos 2x - \cos 4x) \cdot \sin 2x$$

$$= \frac{1}{4} \sin 4x - \frac{1}{4} (\sin 6x - \sin 2x).$$

于是,

$$\int_0^{\frac{\pi}{2}} \sin x \sin 2x \sin 3x dx$$

$$= \left(-\frac{1}{16} \cos 4x + \frac{1}{24} \cos 6x - \frac{1}{8} \cos 2x \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{6}.$$

$$2278. \int_0^{\pi} (x \sin x)^2 dx.$$

解 $\int_0^{\pi} (x \sin x)^2 dx$

$$= \frac{1}{2} \int_0^{\pi} x^2 (1 - \cos 2x) dx$$

$$= \frac{1}{6} x^3 \Big|_0^{\pi} - \frac{1}{2} \int_0^{\pi} x^2 \cos 2x dx$$

$$= \frac{\pi^3}{6} - \frac{x^2}{4} \sin 2x \Big|_0^{\pi} + \frac{1}{2} \int_0^{\pi} x \sin 2x dx$$

$$\begin{aligned}
&= \frac{\pi^2}{6} - \frac{x}{4} \cos 2x \Big|_0^{\pi} + \frac{1}{4} \int_0^{\pi} \cos 2x dx \\
&= \frac{\pi^2}{6} - \frac{\pi}{4}.
\end{aligned}$$

2279. $\int_0^x e^x \cos^2 x dx.$

$$\begin{aligned}
\text{解} \quad \int_0^x e^x \cos^2 x dx &= \int_0^x \frac{e^x(1+\cos 2x)}{2} dx \\
&= \frac{e^x}{2} + \frac{e^x}{10} (\cos 2x + 2\sin 2x)^{*)} \Big|_0^x \\
&= \frac{3}{5}(e^x - 1).
\end{aligned}$$

*) 利用1828题的结果。

2280. $\int_0^{\ln 2} \text{sh}^4 x dx.$

$$\begin{aligned}
\text{解} \quad \int_0^{\ln 2} \text{sh}^4 x dx &= \int_0^{\ln 2} \text{sh}^2 x \cdot (\text{ch}^2 x - 1) dx \\
&= \frac{1}{4} \int_0^{\ln 2} \text{sh}^2 2x dx - \int_0^{\ln 2} \text{sh}^2 x dx \\
&= \frac{1}{4} \left(-\frac{x}{2} + \frac{1}{8} \text{sh} 4x \right)^{*)} \Big|_0^{\ln 2} \\
&\quad - \left(-\frac{x}{2} + \frac{1}{4} \text{sh} 2x \right)^{*)} \Big|_0^{\ln 2} \\
&= \frac{3}{8} \ln 2 - \frac{225}{1024}.
\end{aligned}$$

*) 利用1761题的结果。

利用递推公式来计算下列依赖于取正整数值的参数 n 的积分。

$$2281. I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx.$$

$$\text{解 } I_n = - \int_0^{\frac{\pi}{2}} \sin^{n-1} x d(\cos x)$$

$$= - \sin^{n-1} x \cdot \cos x \Big|_0^{\frac{\pi}{2}}$$

$$+ (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x dx,$$

移项合并得

$$I_n = \frac{n-1}{n} I_{n-2}.$$

利用上述递推公式即可求得

$$I_n = \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & \text{若 } n=2k; \\ \frac{(2k)!!}{(2k+1)!!}, & \text{若 } n=2k+1. \end{cases}$$

$$2282. I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx.$$

解 设 $\frac{\pi}{2} - x = t$, 则 $dx = -dt$, 且

$$\cos x = \cos \left(\frac{\pi}{2} - t \right) = \sin t.$$

代入得

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n} t dt.$$

因此，与2281题的结果相同。

$$2283. I_n = \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n} x dx.$$

$$\begin{aligned} \text{解 } I_n &= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-2} x \cdot (\sec^2 x - 1) dx \\ &= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-2} x d(\operatorname{tg} x) - \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-2} x dx \\ &= \frac{1}{2n-1} - I_{n-1}, \end{aligned}$$

即

$$I_n = \frac{1}{2n-1} - I_{n-1}.$$

由于 $I_0 = \int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4}$ ，于是推得

$$\begin{aligned} I_n &= \frac{1}{2n-1} - \left(\frac{1}{2n-3} - I_{n-2} \right) = \dots \\ &= \frac{1}{2n-1} - \frac{1}{2n-3} + \frac{1}{2n-5} - \dots + (-1)^n I_0 \\ &= (-1)^n \left\{ \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \dots \right) \right\} \end{aligned}$$

$$\left. + \frac{(-1)^{n-1}}{2n-1} \right\}.$$

$$2284. I_n = \int_0^1 (1-x^2)^n dx.$$

解 设 $x = \sin t$, 代入得

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt = \frac{(2n)!!}{(2n+1)!!} \\ &= 2^{2n} \cdot \frac{(n!)^2}{(2n+1)!}. \end{aligned}$$

$$2285. I_n = \int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}.$$

解 设 $x = \sin t$, 代入得

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t dt.$$

因此, 与2281题的结果相同。

$$2286. I_n = \int_0^1 x^n (\ln x)^n dx.$$

$$\begin{aligned} \text{解 } I_n &= \left. -\frac{1}{m+1} x^{n+1} \ln^n x \right|_0^1 \\ &\quad - \frac{n}{m+1} \int_0^1 x^n (\ln x)^{n-1} dx, \end{aligned}$$

于是,

$$\begin{aligned} I_n &= -\frac{n}{m+1} I_{n-1} \\ &= \left(-\frac{n}{m+1}\right) \left(-\frac{n-1}{m+1}\right) \cdots \left(-\frac{1}{m+1}\right) I_0 \end{aligned}$$

$$= (-1)^n \cdot \frac{n!}{(n+1)^{n+1}}.$$

$$2287. I_n = \int_0^{\frac{\pi}{4}} \left(\frac{\sin x - \cos x}{\sin x + \cos x} \right)^{2n+1} dx.$$

$$\text{解 } I_n = \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n+1} \left(x - \frac{\pi}{4} \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-1} \left(x - \frac{\pi}{4} \right) \cdot \left[\sec^2 \left(x - \frac{\pi}{4} \right) - 1 \right] dx$$

$$= \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n-1} \left(x - \frac{\pi}{4} \right) d \left[\operatorname{tg} \left(x - \frac{\pi}{4} \right) \right] - I_{n-1}$$

$$= -\frac{1}{2n} - I_{n-1},$$

即

$$I_n = -\frac{1}{2n} - I_{n-1}.$$

递推之，得

$$I_n = -\frac{1}{2n} + \frac{1}{2(n-1)} - \frac{1}{2(n-2)} + \dots$$

$$+ (-1)^n \cdot \frac{1}{2} + (-1)^n I_0.$$

但

$$I_0 = \int_0^{\frac{\pi}{4}} \operatorname{tg} \left(x - \frac{\pi}{4} \right) dx = -\ln \left| \cos \left(x - \frac{\pi}{4} \right) \right| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln \frac{\sqrt{2}}{2} = -\ln \sqrt{2},$$

于是,

$$I_n = (-1)^n \left\{ -\ln \sqrt{2} + \frac{1}{2} \left[1 - \frac{1}{2} + \dots \right. \right. \\ \left. \left. + (-1)^{n-1} \frac{1}{n} \right] \right\}.$$

设 $f(x) = f_1(x) + if_2(x)$ 是实变数 x 的复函数,
其中 $f_1(x) = \operatorname{Re} f(x)$, $f_2(x) = \operatorname{Im} f(x)$ 及 $i = \sqrt{-1}$,
则按定义有:

$$\int f(x) dx = \int f_1(x) dx + i \int f_2(x) dx.$$

显而易见

$$\operatorname{Re} \int f(x) dx = \int \operatorname{Re} f(x) dx.$$

$$\text{及 } \operatorname{Im} \int f(x) dx = \int \operatorname{Im} f(x) dx.$$

2288. 利用尤拉氏公式

$$e^{ix} = \cos x + i \sin x,$$

证明:

$$\int_0^{2\pi} e^{inx} e^{-imx} dx = \begin{cases} 0, & \text{若 } m \neq n, \\ 2\pi, & \text{若 } m = n. \end{cases}$$

(n 及 m 为整数).

证 当 $m = n$ 时,

$$\int_0^{2\pi} e^{inx} e^{-inx} dx = \int_0^{2\pi} dx = 2\pi.$$

当 $m \neq n$ 时,

$$\int_0^{2\pi} e^{inx} e^{-imx} dx$$

$$\begin{aligned}
 &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx) dx \\
 &= \int_0^{2\pi} \cos(m-n)x dx - i \int_0^{2\pi} \sin(m-n)x dx = 0.
 \end{aligned}$$

2289. 证明

$$\int_a^b e^{(\alpha+i\beta)x} dx = \frac{e^{b(\alpha+i\beta)} - e^{a(\alpha+i\beta)}}{\alpha+i\beta}$$

(α 及 β 为常数)

证 $\int_a^b e^{(\alpha+i\beta)x} dx$

$$= \int_a^b e^{\alpha x} \cos \beta x dx + i \int_a^b e^{\alpha x} \sin \beta x dx$$

$$= \frac{e^{\alpha x} (\alpha \cos \beta x + \beta \sin \beta x + i(\alpha \sin \beta x - \beta \cos \beta x))}{\alpha^2 + \beta^2} \Big|_a^b$$

$$= \frac{e^{\alpha x} (\alpha - i\beta)(\cos \beta x + i \sin \beta x)}{(\alpha + i\beta)(\alpha - i\beta)} \Big|_a^b$$

$$= \frac{e^{(\alpha+i\beta)x}}{\alpha+i\beta} \Big|_a^b = \frac{e^{(\alpha+i\beta)b} - e^{(\alpha+i\beta)a}}{\alpha+i\beta}$$

利用尤拉氏公式:

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}),$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}),$$

计算下列积分 (m 及 n 为正整数),

2290. $\int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx$.

解：方法一：记

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx,$$

易见 $I_0 = \frac{1}{4} I$ ，其中

$$I = \int_0^{2\pi} \sin^{2m} x \cos^{2n} x dx,$$

利用尤拉公式，有

$$\begin{aligned} \sin^{2m} x \cos^{2n} x &= \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^{2m} \left(\frac{e^{ix} + e^{-ix}}{2} \right)^{2n} \\ &= \frac{(-1)^m}{2^{2n+2m}} \sum_{l=0}^{2m} (-1)^l c_{2m}^l e^{2(m-l)ix} \sum_{l=0}^{2n} c_{2n}^l e^{2(n-l)ix} \\ &= \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} \sum_{l=0}^{2n} (-1)^k c_{2m}^k c_{2n}^l e^{2(m+n-k-l)ix} \\ &= \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} \sum_{l=0}^{2n} (-1)^k c_{2m}^k c_{2n}^l [\cos 2(m+n-k-l)x + i \sin 2(m+n-k-l)x], \end{aligned}$$

今不妨设 $m \leq n^*$ ，作积分计算，则有

$$I = \frac{(-1)^m}{2^{2n+2m}} \sum_{k=0}^{2m} \sum_{l=0}^{2n} (-1)^k c_{2m}^k c_{2n}^l \left(\int_0^{2\pi} \cos(m+n-k-l)x dx \right)$$

* 若 $m > n$ 作代换 $x = \frac{\pi}{2} - t$ 即得。

$$\begin{aligned}
& -k-l) 2x dx + i \int_0^{2\pi} \sin(m+n-k-l) 2x dx \\
& = \frac{(-1)^m \pi}{2^{2m+2n-1}} \sum_{k+l=m+n} \sum_{\substack{0 < k < 2m \\ 0 < l < 2n}} (-1)^k C_{2m}^k C_{2n}^l \\
& = \frac{(-1)^m \pi}{2^{2m+2n-1}} \sum_{k=0}^{2m} (-1)^k C_{2m}^k C_{2n}^{m+n-k}
\end{aligned}$$

经计算，可以验证有：

$$\begin{aligned}
& (-1)^m \sum_{k=0}^{2m} (-1)^k C_{2m}^k C_{2n}^{m+n-k} \\
& = \frac{(2m)!(2n)!}{m!n!(m+n)!}
\end{aligned}$$

于是得

$$I_0 = \frac{1}{4} I = \frac{\pi(2m)!(2n)!}{2^{2m+2n+1} m!n!(m+n)!}$$

方法二：

令 $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx$ 。显然

$$I_{m,0} = \int_0^{\frac{\pi}{2}} \sin^{2m} x dx = \frac{(2m-1)!!}{(2m)!!} \cdot \frac{\pi}{2}$$

* 用 $C_m^k C_{2n}^{m+n-k} = C_{2n}^m C_{2n}^k (C_{m+n}^m)^{-2} C_{m+n}^k C_{m+n}^{2m-k}$ ，以及由恒等式 $(1-x)^{m+n} (1+x)^{m+n} = (1-x^2)^{m+n}$ 展开，取 x^{2m} 的系数的关系式

$$\sum_{k=0}^{2m} (-1)^k C_{m+n}^k C_{m+n}^{2m-k} = (-1)^m C_{m+n}^m \text{ 可以验证。}$$

$$\begin{aligned}
I_{m, n} &= \int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n-1} x dx \\
&= \frac{1}{2m+1} \int_0^{\frac{\pi}{2}} \cos^{2n-1} x d\sin^{2m+1} x \\
&= \frac{1}{2m+1} \cos^{2n-1} x \sin^{2m+1} x \Big|_0^{\frac{\pi}{2}} \\
&\quad - \frac{1}{2m+1} \int_0^{\frac{\pi}{2}} \sin^{2m+1} x d\cos^{2n-1} x \\
&= \frac{2n-1}{2m+1} \int_0^{\frac{\pi}{2}} \sin^{2m+2} x \cos^{2n-2} x dx \\
&= \frac{2n-1}{2m+1} \int_0^{\frac{\pi}{2}} \sin^{2m} x (1 - \cos^2 x) \cos^{2n-2} x dx \\
&= \frac{2n-1}{2m+1} I_{m, n-1} - \frac{2n-1}{2m+1} I_{m, n}
\end{aligned}$$

整理后得

$$I_{m, n} = \frac{2n-1}{2(m+n)} I_{m, n-1}$$

由此不难得到

$$\begin{aligned}
I_{m, n} &= \frac{(2n-1)!!}{2^n(m+n)(m+n-1)\cdots(m+1)} I_{m, 0} \\
&= \frac{(2n-1)!! m!}{2^n(m+n)!} \cdot \frac{(2m-1)!!}{(2m)!} \cdot \frac{\pi}{2} \\
&= \frac{\pi(2n-1)!!(2m-1)!!}{2^{n+m+1}(m+n)!}
\end{aligned}$$

$$= \frac{\pi(2m)!(2n)!}{2^{2m+2n+1} m!n!(m+n)!}.$$

2291. $\int_0^\pi \frac{\sin nx}{\sin x} dx.$

解 设 $u = \frac{\sin nx}{\sin x}$, 利用尤拉公式得

$$u = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}}.$$

当 $n=2k$ 时,

$$\begin{aligned} u &= (e^{ikx} + e^{-ikx}) \cdot (e^{i(k-1)x} + e^{i(k-2)x} + \dots \\ &\quad + e^{-i(k-3)x} + e^{-i(k-1)x}) \\ &= e^{i(2k-1)x} + e^{i(2k-3)x} + \dots + e^{ix} + e^{-ix} \\ &\quad + \dots + e^{-i(2k-1)x} \\ &= 2[\cos(2k-1)x + \cos(2k-3)x + \dots + \cos x], \end{aligned}$$

于是,

$$\begin{aligned} \int_0^\pi u dx &= 2 \left[\frac{\sin(2k-1)x}{2k-1} + \frac{\sin(2k-3)x}{2k-3} \right. \\ &\quad \left. + \dots + \sin x \right] \Big|_0^\pi = 0. \end{aligned}$$

当 $n=2k+1$ 时, 同上得

$$u = 2[\cos 2kx + \cos 2(k-1)x + \dots + \cos 2x] + 1.$$

于是,

$$\int_0^\pi u dx = \pi.$$

最后得到

$$\int_0^{\pi} \frac{\sin nx}{\sin x} dx = \begin{cases} 0, & n \text{ 为偶数;} \\ \pi, & n \text{ 为奇数.} \end{cases}$$

2292. $\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x} dx.$

解 $\frac{\cos(2n+1)x}{\cos x} = \frac{e^{i(2n+1)x} + e^{-i(2n+1)x}}{e^{ix} + e^{-ix}}$

$$= e^{2nix} - e^{2(n-1)ix} + \dots + (-1)^n + \dots + e^{-2nix}$$

$$= 2[\cos 2nx - \cos 2(n-1)x + \dots$$

$$+ (-1)^{n-1} \cos 2x] + (-1)^n.$$

于是,

$$\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \pi.$$

2293. $\int_0^{\pi} \cos^n x \cos nx dx$

解 $\cos^n x \cos nx = \frac{1}{2^{n+1}} (e^{ix} + e^{-ix})^n (e^{inx} + e^{-inx})$

$$= \frac{1}{2^n} [\cos 2nx + c_n^1 \cos 2(n-1)x + \dots$$

$$+ c_n^{n-1} \cos 2x + 1].$$

于是,

$$\int_0^{\pi} \cos^n x \cos nx dx = \frac{\pi}{2^n}.$$

2294. $\int_0^{\pi} \sin^n x \sin nx dx.$

解 方法一:

$$\begin{aligned}
 & \int_0^{\pi} \sin^n x \sin nx dx \\
 &= \frac{1}{(2i)^{n+1}} \int_0^{\pi} \left[\sum_{k=0}^n (-1)^k C_n^k e^{i(n-2k)x} (e^{inx} - e^{-inx}) \right] dx \\
 &= \frac{1}{(2i)^{n+1}} \left[\sum_{k=0}^n (-1)^k C_n^k \int_0^{\pi} e^{i(2n-2k)x} dx - \sum_{k=0}^n (-1)^k C_n^k \int_0^{\pi} e^{-i2kx} dx \right] \\
 &= \frac{1}{2^{n+1} i^{n+1}} \left[(-1)^n C_n^n \pi - (-1)^0 C_n^0 \pi \right] \\
 &= \begin{cases} 0, & n \text{ 为偶数;} \\ \frac{\pi}{2^n} \cdot (-1)^{\frac{n+1}{2}}, & n \text{ 为奇数.} \end{cases}
 \end{aligned}$$

由于

$$\sin \frac{n\pi}{2} = \begin{cases} 0, & n \text{ 为偶数,} \\ (-1)^{\frac{n+1}{2}}, & n \text{ 为奇数,} \end{cases}$$

于是,

$$\int_0^{\pi} \sin^n x \sin nx dx = \frac{\pi}{2^n} \sin \frac{n\pi}{2}.$$

方法二:

设 $x = \frac{\pi}{2} - t$, 则

$$\begin{aligned}
& \int_0^{\pi} \sin^n x \sin nx \, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \sin \left(\frac{n\pi}{2} - nt \right) dt \\
&= \sin \frac{n\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \cos nt \, dt \\
&\quad - \cos \frac{n\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \sin nt \, dt \\
&= \sin \frac{n\pi}{2} \int_0^{\pi} \cos^n x \cos nx \, dx \\
&= \frac{\pi}{2^n} \sin \frac{n\pi}{2}.
\end{aligned}$$

求下列积分 (n 为自然数):

2295. $\int_0^{\pi} \sin^{n-1} x \cos(n+1)x \, dx.$

$$\begin{aligned}
& \text{解 } \int_0^{\pi} \sin^{n-1} x \cos(n+1)x \, dx \\
&= \int_0^{\pi} \sin^{n-1} x (\cos nx \cos x - \sin nx \sin x) \, dx \\
&= \int_0^{\pi} \sin^{n-1} x \cos nx \, d(\sin x) - \int_0^{\pi} \sin^n x \sin nx \, dx \\
&= \frac{\sin^n x \cos nx}{n} \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin^n x \, d(\cos nx) \\
&\quad - \int_0^{\pi} \sin^n x \sin nx \, dx \\
&= 0.
\end{aligned}$$

$$2296. \int_0^{\pi} \cos^{n-1} x \sin(n+1)x dx.$$

解 考虑积分

$$I = \int_0^{\pi} \cos^{n+1} x \sin(n+1)x dx,$$

并对它作两次分部积分, 可得

$$I = I - \frac{n}{n+1} \int_0^{\pi} \cos^{n-1} x \sin(n+1)x dx.$$

于是,

$$\int_0^{\pi} \cos^{n-1} x \sin(n+1)x dx = 0.$$

本题也可不用分部积分法. 事实上, $\cos^{n-1} x \cdot \sin(n+1)x$ 是以 π 为周期的函数, 又是奇函数, 于是

$$\begin{aligned} & \int_0^{\pi} \cos^{n-1} x \sin(n+1)x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \sin(n+1)x dx = 0. \end{aligned}$$

$$2297. \int_0^{2\pi} e^{-ax} \cos^{2n} x dx.$$

解 方法一:

$$\cos^{2n} x = \left(\frac{e^{ix} + e^{-ix}}{2} \right)^{2n}$$

$$= \frac{1}{2^{2n}} \left[c_{2n}^n + 2 \sum_{k=0}^{n-1} c_{2n}^k \cos 2(n-k)x \right].$$

于是

$$\begin{aligned}
 I &= \int_0^{2\pi} e^{-ax} \cos^{2n} x dx \\
 &= \frac{1}{2^{2n}} \left\{ c_{2n}^n \int_0^{2\pi} e^{-ax} dx + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
 &\quad \left. \cdot \int_0^{2\pi} e^{-ax} \cos 2(n-k)x dx \right\} \\
 &= \frac{1}{2^{2n}} \left\{ -\frac{1}{a} c_{2n}^n e^{-ax} \Big|_0^{2\pi} + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
 &\quad \left. \cdot \frac{(2n-2k) \sin 2(n-k)x - a \cos 2(n-k)x}{a^2 + (2n-2k)^2} e^{-ax} \Big|_0^{2\pi} \right\} \\
 &= \frac{1}{2^{2n}} \left\{ -\frac{1}{a} c_{2n}^n (e^{-2\pi a} - 1) - a(e^{-2\pi a} - 1) \right. \\
 &\quad \left. \cdot \sum_{k=0}^{n-1} \frac{2c_{2n}^k}{a^2 + (2n-2k)^2} \right\} \\
 &= \frac{1 - e^{-2\pi a}}{2^{2n} \cdot a} \left\{ c_{2n}^n + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
 &\quad \left. \cdot \frac{a^2}{a^2 + (2n-2k)^2} \right\},
 \end{aligned}$$

即

$$\begin{aligned}
 &\int_0^{2\pi} e^{-ax} \cos^{2n} x dx \\
 &= \frac{1 - e^{-2\pi a}}{2^{2n} \cdot a} \cdot \left\{ c_{2n}^n + 2 \sum_{k=0}^{n-1} c_{2n}^k \right. \\
 &\quad \left. \cdot \frac{a^2}{a^2 + (2n-2k)^2} \right\}.
 \end{aligned}$$

方法二:

由于

$$\begin{aligned}\int_0^{2\pi} e^{(a+ik)x} dx &= \frac{e^{(a+ik)x}}{a+ik} \Big|_0^{2\pi} \\ &= \frac{e^{2\pi a} - 1}{a+ik} = \frac{(e^{2\pi a} - 1)(a-ik)}{a^2 + k^2},\end{aligned}$$

取实部, 得

$$\operatorname{Re} \int_0^{2\pi} e^{(a+ik)x} dx = \frac{a(e^{2\pi a} - 1)}{a^2 + k^2}$$

于是,

$$\begin{aligned}& \int_0^{2\pi} e^{-ax} \cos^{2n} x dx \\ &= \int_0^{2\pi} e^{-ax} \left(\frac{e^{ix} + e^{-ix}}{2} \right)^{2n} dx \\ &= \frac{1}{2^{2n}} \int_0^{2\pi} e^{-ax} \left(\sum_{k=0}^{2n} c_{2n}^k e^{i(2n-2k)x} \right) dx \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} c_{2n}^k \int_0^{2\pi} e^{-a+i(2n-2k)x} dx \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} c_{2n}^k \frac{e^{-2\pi a} - 1}{-a+i(2n-2k)} \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} c_{2n}^k \frac{a(1 - e^{-2\pi a})}{a^2 + (2n-2k)^2} \\ &= \frac{1 - e^{-2\pi a}}{2^{2n} \cdot a} \left[c_{2n}^{2n} + 2 \sum_{k=0}^{n-1} c_{2n}^k \frac{a^2}{a^2 + (2n-2k)^2} \right].\end{aligned}$$

$$2298. \int_0^{\frac{\pi}{2}} \ln \cos x \cdot \cos 2nx dx$$

解 利用分部积分得

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \ln \cos x \cdot \cos 2nx dx \\ &= \frac{1}{2n} \sin 2nx \cdot \ln \cos x \Big|_0^{\frac{\pi}{2}} \\ & \quad + \frac{1}{2n} \int_0^{\frac{\pi}{2}} \frac{\sin 2nx \cdot \sin x}{\cos x} dx \\ &= 0^{*}) + \frac{1}{4n} \int_0^{\frac{\pi}{2}} \frac{\cos(2n-1)x}{\cos x} dx \\ & \quad - \frac{1}{4n} \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} dx. \end{aligned}$$

对于上述等式右端的第二项和第三项的被积函数有下列等式:

$$\begin{aligned} \frac{\cos(2n-1)x}{\cos x} &= \frac{e^{i(2n-1)x} + e^{-i(2n-1)x}}{e^{ix} + e^{-ix}} \\ &= 2[\cos 2(n-1)x - \cos 2(n-2)x + \dots \\ & \quad + (-1)^{n-2} \cos 2x] + (-1)^{n-1}, \\ \frac{\cos(2n+1)x}{\cos x} &= 2[\cos 2nx - \cos 2(n-1)x + \dots \\ & \quad + (-1)^{n-1} \cos 2x] + (-1)^n. \end{aligned}$$

由于积分

$$\int_0^{\frac{\pi}{2}} \cos 2kx dx \quad (k \text{ 为任意的正整数})$$

的值恒等于零，所以积分

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2n-1)x}{\cos x} dx \quad \text{及} \quad \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} dx$$

分别等于 $(-1)^{n-1} \frac{\pi}{2}$ 及 $(-1)^n \frac{\pi}{2}$ 。

这样，我们得到

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \ln \cos x \cdot \cos 2nx dx \\ &= \frac{1}{4n} \left[(-1)^{n-1} \frac{\pi}{2} - (-1)^n \frac{\pi}{2} \right] \\ &= \frac{\pi}{4n} (-1)^{n-1}. \end{aligned}$$

*) 在 $x=0$ 处， $\sin 2nx \cdot \ln \cos x = 0$ ；而在 $x = \frac{\pi}{2}$

处，为“ $0 \cdot \infty$ ”型，采用洛比塔法则定值：

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}-0} \sin 2nx \cdot \ln \cos x &= \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\ln \cos x}{\frac{1}{\sin 2nx}} \\ &= \frac{1}{2n} \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\sin x \cdot \sin^2 2nx}{\cos x \cdot \cos 2nx} \\ &= \frac{1}{2n} \lim_{x \rightarrow \frac{\pi}{2}-0} \frac{\cos x \cdot \sin^2 2nx + 4n \sin x \sin 2nx \cos 2nx}{-\sin x \cos 2nx - 2n \cos x \sin 2nx} \\ &= 0. \end{aligned}$$

2299. 利用多次的部分积分法, 计算尤拉氏积分:

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx,$$

式中 m 及 n 为正整数.

$$\begin{aligned} \text{解 } B(m, n) &= \frac{1}{m} x^m (1-x)^{n-1} \Big|_0^1 \\ &\quad + \frac{n-1}{m} \int_0^1 x^m (1-x)^{n-2} dx \\ &= \frac{n-1}{m} B(m+1, n-1). \end{aligned}$$

继续利用部分积分法, 可得

$$\begin{aligned} B(m, n) &= \frac{(n-1)(n-2)\cdots 2 \cdot 1}{m(m+1)\cdots(m+n-2)} \int_0^1 x^{m+n-2} dx \\ &= \frac{(n-1)!(m-1)!}{(m+n-2)!} \\ &\quad \cdot \frac{1}{m+n-1} x^{m+n-1} \Big|_0^1 \\ &= \frac{(n-1)!(m-1)!}{(m+n-1)!}. \end{aligned}$$

2300. 勒让德多项式 $P_n(x)$ 被下面公式来定义:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n] \quad (n=0, 1, 2, \dots).$$

证明

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0, & \text{若 } m \neq n, \\ \frac{2}{2n+1}, & \text{若 } m = n. \end{cases}$$

证：当 $m \neq n$ 时，不失一般性，设 $n < m$ 。由于 $P_n(x)$ 为 n 次的多项式，我们记

$$P_m(x) = R^{(m)}(x),$$

$$\text{其中 } R(x) = \frac{1}{2^n n!} (x^2 - 1)^n.$$

利用多次部分积分法得

$$\begin{aligned} & \int_{-1}^1 P_m(x) P_n(x) dx \\ &= \left[P_n(x) R^{(m-1)}(x) - P_n'(x) R^{(m-2)}(x) + \dots \right. \\ & \quad \left. + (-1)^{m-1} P_n^{(m-1)}(x) R(x) \right] \Big|_{-1}^1 \\ & \quad + (-1)^m \int_{-1}^1 R(x) P_n^{(n)}(x) dx = 0. \end{aligned}$$

当 $m = n$ 时，

$$\begin{aligned} & \int_{-1}^1 P_n(x) P_n(x) dx \\ &= \frac{1}{2^{2n} (n!)^2} \int_{-1}^1 \left[\frac{d^n (x^2 - 1)^n}{dx^n} \right]^2 dx, \end{aligned}$$

设 $u = \frac{d^n}{dx^n} [(x^2 - 1)^n]$, $v = (x^2 - 1)^n$, 则

$$\begin{aligned} & \int_{-1}^1 P_n^2(x) dx \\ &= \frac{1}{2^{2n} (n!)^2} \left[uv^{(n-1)} - u'v^{(n-2)} + \dots \right. \\ & \quad \left. + (-1)^{n-1} u^{(n-1)} v \right] \Big|_{-1}^1 + (-1)^n \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{2n}(n!)^2} \int_{-1}^1 v u^{(n)} dx \\
&= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^1 (x^2-1)^n \cdot \frac{d^{2n}}{dx^{2n}} [(x^2-1)^n] dx \\
&= \frac{(2n)!}{2^{2n-1}(n!)^2} \int_0^1 (1-x^2)^n dx \\
&= \frac{(2n)!}{2^{2n-1}(n!)^2} \int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt \\
&= \frac{(2n)!}{2^{2n-1}(n!)^2} \cdot \frac{(2n)!!}{(2n+1)!!}^{**)} = \frac{2}{2n+1}.
\end{aligned}$$

*) 设 $x = \sin t$.

***) 利用2282题的结果.

2301. 设函数 $f(x)$ 在 $[a, b]$ 上可积分, 函数 $F(x)$ 在 $[a, b]$ 内除了有限个点 $c_i (i=1, \dots, p)$ 及点 a 与 b 外皆有 $F'(x) = f(x)$, 而在这除去的有限个点上 $F(x)$ 有第一类的间断点 (广义原函数). 证明

$$\begin{aligned}
\int_a^b f(x) dx &= F(b-0) - F(a+0) \\
&\quad - \sum_{i=1}^p [F(c_i+0) - F(c_i-0)].
\end{aligned}$$

证 为确定起见, 设 $a < c_1 < c_2 < \dots < c_p < b$, 并记 $a = c_0, b = c_{p+1}$. 由于 $f(x)$ 在 $[a, b]$ 上可积, 故

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0^+} \sum_{i=0}^p \int_{c_i+h}^{c_{i+1}-h} f(x) dx.$$

显然, 在 $[c_i+h, c_{i+1}-h]$ 上 $F'(x) = f(x)$, 从而可

应用牛顿—莱布尼兹公式，得

$$\int_{c_i+\eta}^{c_{i+1}-\eta} f(x) dx = F(c_{i+1}-\eta) - F(c_i+\eta),$$

由此可知

$$\begin{aligned} \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} \sum_{i=0}^{p-1} (F(c_{i+1}-\eta) - F(c_i+\eta)) \\ &= \sum_{i=0}^{p-1} (F(c_{i+1}-0) - F(c_i+0)) \\ &= F(b-0) - F(a+0) - \sum_{i=1}^p (F(c_i+0) \\ &\quad - F(c_i-0)). \end{aligned}$$

2302. 设函数 $f(x)$ 在闭区间 $[a, b]$ 上可积分，而

$$F(x) = C + \int_a^x f(\xi) d\xi$$

为 $f(x)$ 的不定积分。证明函数 $F(x)$ 连续且在函数 $f(x)$ 连续的一切点处有等式

$$F'(x) = f(x)$$

成立，问在函数 $f(x)$ 不连续点处函数 $F(x)$ 的导函数为何？

解 由于 $f(x)$ 在 $[a, b]$ 上可积，故必有界： $|f(x)| \leq M$ ($a \leq x \leq b$)。因此，对任何 $x \in [a, b]$ ，有

$$\begin{aligned} &|F(x+\Delta x) - F(x)| \\ &= \left| \int_x^{x+\Delta x} f(\xi) d\xi \right| \leq M \cdot |\Delta x| \rightarrow 0 \quad (\text{当 } \Delta x \rightarrow 0 \text{ 时}). \end{aligned}$$

由此可知 $F(x)$ 在 (a, b) 上连续。

现设 $f(\xi)$ 在点 $\xi = x$ 处连续, 于是, 任给 $\varepsilon > 0$, 存在 $\delta > 0$, 使当 $|\xi - x| < \delta$ 时, 恒有 $|f(\xi) - f(x)| < \varepsilon$.

于是, 当 $0 < |\Delta x| < \delta$ 时, 恒有

$$\begin{aligned} & \left| \frac{F(x + \Delta x) - F(x)}{\Delta x} - f(x) \right| \\ &= \left| \frac{1}{\Delta x} \int_x^{x + \Delta x} [f(\xi) - f(x)] d\xi \right| \\ &\leq \frac{1}{|\Delta x|} \varepsilon \cdot |\Delta x| = \varepsilon, \end{aligned}$$

故 $F'(x)$ 存在, 且

$$F'(x) = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x).$$

而在不连续点处 $F'(x)$ 可能存在也可能不存在.

例如, 设

$$f(x) = \begin{cases} 1, & \text{当 } x = \frac{1}{n}, \\ 0, & \text{当 } x \neq \frac{1}{n}, \end{cases} \quad (n=1, 2, 3, \dots).$$

则 $f(x)$ 在 $[0, 1]$ 的可积性可仿2194题证明, 而且显然有

$$\int_0^x f(t) dt = 0 \quad (0 \leq x \leq 1).$$

然而在点 $x = \frac{1}{n}$ 处, $F(x) = C$ 的导函数 $F'(x)$

$= 0$ 是存在的。

但函数 $f(x) = \operatorname{sgn} x$, 它在 $[-1, 1]$ 上是可积的, 且

$$\int_0^x f(x) dx = |x|,$$

然而在点 $x = 0$ 处, $F(x) = |x| + C$ 的导函数 $F'(x)$ 不存在。

求下列有界非连续函数的不定积分:

2303. $\int \operatorname{sgn} x dx.$

解 $\int \operatorname{sgn} x dx = \int_0^x \operatorname{sgn} x dx + C = |x| + C.$

2304. $\int \operatorname{sgn}(\sin x) dx.$

解 由于 $\operatorname{sgn}(\sin x)$ 在任何有限区间上都可积, 故其

原函数 $F(x) = \int_0^x \operatorname{sgn}(\sin t) dt$ 是 $(-\infty, +\infty)$ 上

的连续函数。对任何 x , 必存在唯一的整数 k 使 $k\pi \leq x < (k+1)\pi$ 。于是

$$\begin{aligned} F(x) &= \int_0^x \operatorname{sgn}(\sin t) dt \\ &= \int_0^{k\pi + \frac{\pi}{2}} \operatorname{sgn}(\sin t) dt + \int_{k\pi + \frac{\pi}{2}}^x \operatorname{sgn}(\sin t) dt \\ &= \frac{\pi}{2} + \int_{k\pi + \frac{\pi}{2}}^x \frac{\sin t}{\sqrt{1 - \cos^2 t}} dt \end{aligned}$$

$$= \frac{\pi}{2} + \arccos(\cos t) \Big|_{t=\frac{\pi}{2}}^x$$

$$= \frac{\pi}{2} + \arccos(\cos x) - \frac{\pi}{2}$$

$$= \arccos(\cos x).$$

故

$$\int \operatorname{sgn}(\sin x) dx = \arccos(\cos x) + C$$

$$(-\infty < x < +\infty).$$

2305. $\int [x] dx (x \geq 0).$

解 $\int [x] dx = C + \int_0^x [x] dx$

$$= C + \sum_{k=0}^{[x]-1} \int_k^{k+1} k dx + \int_{[x]}^x [x] dx$$

$$= C + \sum_{k=0}^{[x]-1} k + [x](x - [x])$$

$$= x \cdot [x] - \frac{[x]^2 + [x]}{2} + C.$$

2306. $\int x[x] dx (x \geq 0).$

解 $\int x[x] dx = C + \int_0^x x[x] dx$

$$= \sum_{k=0}^{[x]-1} \int_k^{k+1} k t dt + \int_{[x]}^x [x] t dt + C$$

$$\begin{aligned}
&= \sum_{k=0}^{[x]-1} \left(\frac{k^2}{2} \Big|_k^{k+1} \right) + \frac{[x]t^2}{2} \Big|_{[x]}^x + C \\
&= \sum_{k=0}^{[x]-1} \left(k^2 + \frac{k}{2} \right) + \frac{[x](x^2 - [x]^2)}{2} + C \\
&= \frac{([x]-1)[x](2[x]-1)}{6} + \frac{[x]([x]-1)}{4} \\
&\quad + \frac{x^2[x] - [x]^3}{2} + C \\
&= \frac{x^2[x]}{2} \\
&\quad - \frac{6[x]^3 - 3[x]([x]-1) - 2[x]([x]-1)(2[x]-1)}{12} \\
&\quad + C \\
&= \frac{x^2[x]}{2} - \frac{[x]([x]+1)(2[x]+1)}{12} + C.
\end{aligned}$$

2307. $\int (-1)^{[x]} dx.$

解 $\int (-1)^{[x]} dx = \int_0^x \operatorname{sgn}(\sin \pi x) dx + C$

$$\begin{aligned}
&= \frac{1}{\pi} \arccos(\cos \pi x) \Big|_0^{x^*} + C \\
&= \frac{1}{\pi} \arccos(\cos \pi x) + C.
\end{aligned}$$

*) 利用2304题的结果.

2308. $\int_0^x f(x) dx$, 其中 $f(x) = \begin{cases} 1, & \text{若 } |x| < 1, \\ 0, & \text{若 } |x| > 1. \end{cases}$

$$\text{解 } \int_0^x f(x) dx = \int_0^1 f(x) dx + \int_1^x f(x) dx$$

$$= \int_0^1 1 \cdot dx + \int_0^x 0 dx = 1 \quad (x \geq 1);$$

$$\int_0^x f(x) dx = \int_0^x 1 \cdot dx = x \quad (|x| < 1),$$

$$\int_0^x f(x) dx$$

$$= - \int_x^{-1} f(x) dx - \int_{-1}^0 f(x) dx = -1 \quad (x \leq -1).$$

合并得

$$\int_0^x f(x) dx = \frac{1}{2}(|1+x| - |1-x|).$$

计算下列有界非连续函数的定积分:

$$2309. \int_0^3 \text{sgn}(x-x^3) dx.$$

$$\text{解 } \text{sgn}(x-x^3) = \begin{cases} 1, & \text{当 } x \in (0, 1) \text{ 时,} \\ -1, & \text{当 } x \in (1, 3) \text{ 时.} \end{cases}$$

于是,

$$\int_0^3 \text{sgn}(x-x^3) dx = \int_0^1 dx - \int_1^3 dx = -1.$$

$$2310. \int_0^2 (e^x) dx.$$

$$\text{解 } \int_0^2 (e^x) dx$$

$$\begin{aligned}
&= \int_0^{\ln 2} 1 dx + \int_{\ln 2}^{\ln 3} 2 \cdot dx + \int_{\ln 3}^{\ln 4} 3 \cdot dx \\
&\quad + \cdots + \int_{\ln 7}^2 7 \cdot dx \\
&= \ln 2 + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + \cdots \\
&\quad + 7(-\ln 7 + 2) \\
&= 14 - (\ln 2 + \ln 3 + \ln 4 + \cdots + \ln 7) \\
&= 14 - \ln 7! .
\end{aligned}$$

2311. $\int_0^6 [x] \sin \frac{\pi x}{6} dx.$

$$\begin{aligned}
\text{解} \quad &\int_0^6 [x] \sin \frac{\pi x}{6} dx \\
&= \int_1^2 \sin \frac{\pi x}{6} dx + \int_2^3 2 \sin \frac{\pi x}{6} dx + \cdots \\
&\quad + \int_5^6 5 \sin \frac{\pi x}{6} dx \\
&= \frac{30}{\pi} .
\end{aligned}$$

2312. $\int_0^{\pi} x \operatorname{sgn}(\cos x) dx.$

$$\begin{aligned}
\text{解} \quad &\int_0^{\pi} x \operatorname{sgn}(\cos x) dx \\
&= \int_0^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\pi} (-x) dx = -\frac{\pi^2}{4} .
\end{aligned}$$

2313. $\int_1^{n+1} \ln(x) dx$, 其中 n 为自然数.

$$\begin{aligned}
 \text{解} \quad & \int_1^{n+1} \ln(x) dx \\
 &= \int_2^3 \ln 2 dx + \int_3^4 \ln 3 dx + \cdots + \int_n^{n+1} \ln n dx \\
 &= \ln n! .
 \end{aligned}$$

2314. $\int_0^1 \operatorname{sgn}(\sin(\ln x)) dx$.

$$\begin{aligned}
 \text{解} \quad & \int_0^1 \operatorname{sgn}(\sin(\ln x)) dx \\
 &= \int_{e^{-\pi}}^1 (-1) dx + \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k+1} \int_{e^{-(k+1)\pi}}^{e^{-k\pi}} dx \\
 &= -1 + 2e^{-\pi} \cdot \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} e^{-(k-1)\pi} \\
 &= -1 + \frac{2e^{-\pi}}{1+e^{-\pi}} = \frac{e^{-\pi}-1}{e^{-\pi}+1} = -\operatorname{th} \frac{\pi}{2}.
 \end{aligned}$$

2315. 求 $\int_E |\cos x| \sqrt{\sin x} dx$, 其中 E 为闭区间 $[0, 4\pi]$ 中使被积分式有意义的一切值所成之集合.

$$\begin{aligned}
 \text{解} \quad & \int_E |\cos x| \sqrt{\sin x} dx \\
 &= \int_0^{\pi} |\cos x| \sqrt{\sin x} dx + \int_{2\pi}^{3\pi} |\cos x| \sqrt{\sin x} dx \\
 &= \int_0^{\frac{\pi}{2}} \cos x \sqrt{\sin x} dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \sqrt{\sin x} dx \\
 &\quad + \int_{2\pi}^{\frac{5\pi}{2}} \cos x \sqrt{\sin x} dx
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{5\pi}{2}}^{\frac{3\pi}{2}} (-\cos x) \sqrt{\sin x} dx \\
& = 4 \int_0^{\frac{\pi}{2}} \cos x \sqrt{\sin x} dx = \frac{8}{3} (\sin x)^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} = \frac{8}{3}.
\end{aligned}$$

§3. 中值定理

1° 函数的平均值 数

$$M[f] = \frac{1}{b-a} \int_a^b f(x) dx$$

称为函数 $f(x)$ 在区间 $[a, b]$ 上的平均值。

若函数 $f(x)$ 在 $[a, b]$ 上连续, 则可求得一点 $c \in (a, b)$ 适合

$$M[f] = f(c).$$

2° 第一中值定理 若: (1) 函数 $f(x)$ 和 $\varphi(x)$ 于闭区间 $[a, b]$ 上有界并可积分; (2) 当 $a < x < b$ 时, 函数 $\varphi(x)$ 不变号, 则

$$\int_a^b f(x) \varphi(x) dx = \mu \int_a^b \varphi(x) dx,$$

式中 $m \leq \mu \leq M$ 及 $M = \sup_{a < x < b} f(x)$, $m = \inf_{a < x < b} f(x)$; (3) 除此而外, 若函数 $f(x)$ 于闭区间 $[a, b]$ 上连续, 则 $\mu = f(c)$, 其中 $a \leq c \leq b$ (编者注: 可以证明, c 可取值使 $a < c < b$)。

3° 第二中值定理 若: (1) 函数 $f(x)$ 和 $\varphi(x)$ 于闭区间 $[a, b]$ 上有界并可积分; (2) 当 $a < x < b$ 时, 函数 $\varphi(x)$ 是单调的, 则

$$\int_a^b f(x)\varphi(x)dx$$

$$= \varphi(a+0) \int_a^\xi f(x)dx + \varphi(b-0) \int_\xi^b f(x)dx,$$

式中 $a \leq \xi \leq b$; (3) 除此而外, 若函数 $\varphi(x)$ 单调下降 (广义的) 且不为负, 则

$$\int_a^b f(x)\varphi(x)dx = \varphi(a+0) \int_a^\xi f(x)dx \quad (a \leq \xi \leq b);$$

(3') 若函数 $\varphi(x)$ 单调上升 (广义的) 且不为负, 则

$$\int_a^b f(x)\varphi(x)dx = \varphi(b-0) \int_\xi^b f(x)dx \quad (a \leq \xi \leq b).$$

2316. 确定下列定积分的符号;

$$(a) \int_0^{2\pi} x \sin x dx,$$

$$(b) \int_0^{2\pi} \frac{\sin x}{x} dx;$$

$$(B) \int_{-2}^2 x^3 2^x dx;$$

$$(r) \int_{\frac{1}{2}}^1 x^2 \ln x dx.$$

解 (a) $\int_0^{2\pi} x \sin x dx$

$$= \int_0^\pi x \sin x dx + \int_\pi^{2\pi} x \sin x dx$$

$$= \int_0^\pi x \sin x dx - \int_0^\pi (t+\pi) \sin t dt$$

$$= -\pi \int_0^\pi \sin x dx < 0.$$

(6) 由第一中值定理知

$$\begin{aligned}
 & \int_0^{2\pi} \frac{\sin x}{x} dx \\
 &= \int_0^{\pi} \frac{\sin x}{x} dx + \int_{\pi}^{2\pi} \frac{\sin x}{x} dx \\
 &= \int_0^{\pi} \frac{\sin x}{x} dx - \int_0^{\pi} \frac{\sin t}{t+\pi} dt \\
 &= \pi \int_0^{\pi} \frac{\sin x}{x(x+\pi)} dx \\
 &= \frac{\pi^2 \sin c}{c(c+\pi)} > 0,
 \end{aligned}$$

其中 $0 < c < \pi$.

(B) 由第一中值定理知

$$\begin{aligned}
 & \int_{-2}^2 x^3 e^x dx \\
 &= \int_{-2}^0 x^3 e^x dx + \int_0^2 x^3 e^x dx \\
 &= \int_2^0 t^3 e^{-t} dt + \int_0^2 x^3 e^x dx \\
 &= \int_0^2 x^3 (e^x - e^{-x}) dx = 2c^3 (e^c - e^{-c}) > 0,
 \end{aligned}$$

其中 $0 < c < 2$.

$$\begin{aligned}
 (\Gamma) \quad & \int_{\frac{1}{2}}^1 x^2 \ln x dx \\
 &= \frac{1}{2} c^2 \ln c < 0 \quad (\text{其中 } \frac{1}{2} < c < 1)
 \end{aligned}$$

2317. 于下列各题中确定那个积分较大:

$$(a) \int_0^{\frac{\pi}{2}} \sin^{10} x dx \text{ 或 } \int_0^{\frac{\pi}{2}} \sin^2 x dx?$$

$$(b) \int_0^1 e^{-x} dx \text{ 或 } \int_0^1 e^{-x^2} dx?$$

$$(B) \int_0^{\pi} e^{-x^2} \cos^2 x dx \text{ 或 } \int_{\pi}^{2\pi} e^{-x^2} \cos^2 x dx?$$

解 (a) 当 $x \in (0, \frac{\pi}{2})$ 时, $0 < \sin x < 1$ 从而

$$0 < \sin^{10} x < \sin^2 x,$$

于是

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx < \int_0^{\frac{\pi}{2}} \sin^2 x dx.$$

(b) 当 $0 < x < 1$ 时, $x > x^2$, 从而

$$e^{-x} < e^{-x^2},$$

于是

$$\int_0^1 e^{-x} dx < \int_0^1 e^{-x^2} dx.$$

$$(B) \int_{\pi}^{2\pi} e^{-x^2} \cos^2 x dx$$

$$= \int_0^{\pi} e^{-(\pi+x)^2} \cos^2 x dx < \int_0^{\pi} e^{-x^2} \cos^2 x dx.$$

2318. 求下列已知函数在所给区间内的平均值:

(a) $f(x) = x^2$ 在 $[0, 1]$ 上;

(b) $f(x) = \sqrt{x}$ 在 $[0, 100]$ 上;

(B) $f(x) = 10 + 2\sin x + 3\cos x$ 在 $(0, 2\pi]$ 上;

(Г) $f(x) = \sin x \sin(x + \varphi)$ 在 $(0, 2\pi]$ 上。

解 (a) $M(f) = \int_0^1 x^2 dx = \frac{1}{3};$

(б) $M(f) = \frac{1}{100} \int_0^{100} \sqrt{x} dx = 6\frac{2}{3};$

(B) $M(f) = \frac{1}{2\pi} \int_0^{2\pi} (10 + 2\sin x + 3\cos x) dx$
 $= 10;$

(Г) $M(f) = \frac{1}{2\pi} \int_0^{2\pi} \sin x \cdot \sin(x + \varphi) dx$
 $= \frac{1}{2} \cos \varphi.$

2319. 求椭圆之焦径

$$r = \frac{p}{1 - \varepsilon \cos \varphi} \quad (0 < \varepsilon < 1)$$

之长的平均值。

解 设 $\varphi = \pi + t$, 则

$$\begin{aligned} M(r) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{p}{1 - \varepsilon \cos \varphi} d\varphi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{p}{1 + \varepsilon \cos t} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{p}{1 + \varepsilon \cos \varphi} d\varphi \\ &= \frac{p}{2\pi} \frac{2\pi}{\sqrt{1 - \varepsilon^2}} \quad *) \end{aligned}$$

$$= \frac{b}{\sqrt{1-\varepsilon^2}} = b,$$

其中 b 为椭圆的短半轴.

*) 利用2213题的结果.

2320. 求初速度为 v_0 之自由落体的速度之平均值.

解 自由落体的速度为 $v = v_0 + gt$, 从 $t = 0$ 到 $t = T$ 时间内的速度的平均值

$$\begin{aligned} M(v) &= \frac{1}{T} \int_0^T (v_0 + gt) dt = \frac{1}{2} gT + v_0 \\ &= \frac{1}{2} (v_0 + v_T). \end{aligned}$$

物理意义: 平均速度等于初速与末速之和的一半.

2321. 电流的强度依下面的规律变化

$$i = i_0 \sin\left(\frac{2\pi t}{T} + \varphi\right),$$

其中 i_0 表振幅, t 表时间, T 表周期, φ 表初相, 求电流强度之平方的平均值.

$$\begin{aligned} \text{解 } M(i^2) &= \frac{1}{T} \int_0^T i_0^2 \sin^2\left(\frac{2\pi t}{T} + \varphi\right) dt \\ &= \frac{i_0^2}{2\pi} \left[\frac{1}{2} \left(\frac{2\pi t}{T} + \varphi\right) \right. \\ &\quad \left. - \frac{1}{4} \sin 2\left(\frac{2\pi t}{T} + \varphi\right) \right] \Big|_0^T = \frac{i_0^2}{2}. \end{aligned}$$

将上式开平方, 即得电流的有效值 $\frac{i_0}{\sqrt{2}}$.

2322. 命 $\int_0^x f(t)dt = xf(\theta x)$, 求 θ , 设:

(a) $f(t) = t^n$ ($n > -1$); (b) $f(t) = \ln t$;

(B) $f(t) = e^t$,

$\lim_{x \rightarrow 0} \theta$ 及 $\lim_{x \rightarrow +\infty} \theta$ 等于甚么?

解 (a) $\int_0^x f(t)dt = \int_0^x t^n dt = \frac{x^{n+1}}{n+1}$, 从而

$$\frac{x^{n+1}}{n+1} = \theta^n x^{n+1}.$$

于是

$$\theta = \sqrt[n]{\frac{1}{n+1}}.$$

(b) $\int_0^x f(t)dt = \int_0^x \ln t dt = t(\ln t - 1) \Big|_0^x$
 $= x(\ln x - 1),$

从而

$$x(\ln x - 1) = x \ln \theta x,$$

于是

$$\theta = \frac{1}{e}.$$

(B) $\int_0^x f(t)dt = \int_0^x e^t dt = e^t \Big|_0^x = e^x - 1$, 从而
 $e^x - 1 = xe^{\theta x},$

于是

$$\theta = \frac{1}{x} \ln \frac{e^x - 1}{x}.$$

由于 $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ ，故当 $x \rightarrow 0$ 时， $\frac{1}{x} \ln \frac{e^x - 1}{x}$ 是

$\frac{0}{0}$ 型未定形。因此

$$\begin{aligned} \lim_{x \rightarrow 0} \theta &= \lim_{x \rightarrow 0} \frac{1}{x} \ln \frac{e^x - 1}{x} \\ &= \lim_{x \rightarrow 0} \left[\frac{x}{e^x - 1} \cdot \frac{xe^x - (e^x - 1)}{x^2} \right] \\ &= \lim_{x \rightarrow 0} \frac{xe^x - e^x + 1}{x(e^x - 1)} = \lim_{x \rightarrow 0} \frac{xe^x}{e^x - 1 + xe^x} \\ &= \lim_{x \rightarrow 0} \frac{1}{e^{-x} \frac{e^x - 1}{x} + 1} = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow +\infty} \theta &= \lim_{x \rightarrow +\infty} \frac{1}{x} \ln \frac{e^x - 1}{x} = \lim_{x \rightarrow +\infty} \frac{xe^x - e^x + 1}{x(e^x - 1)} \\ &= \lim_{x \rightarrow +\infty} \frac{1 - \frac{1}{x} + \frac{1}{xe^x}}{1 - \frac{1}{e^x}} = 1, \end{aligned}$$

于是，

$$\lim_{x \rightarrow 0} \theta = \frac{1}{2} \text{ 及 } \lim_{x \rightarrow +\infty} \theta = 1.$$

利用第一中值定理估计积分：

2323. $\int_0^{2\pi} \frac{dx}{1 + 0.5 \cos x}$

解 由于

$$\frac{1}{1+0.5} \leq \frac{1}{1+0.5\cos x} \leq \frac{1}{1-0.5},$$

即

$$\frac{2}{3} \leq \frac{1}{1+0.5\cos x} \leq 2.$$

于是

$$\frac{4\pi}{3} \leq \int_0^{2\pi} \frac{dx}{1+0.5\cos x} \leq 4\pi,$$

即

$$\int_0^{2\pi} \frac{dx}{1+0.5\cos x} = \frac{8\pi}{3} \pm \frac{4\pi}{3}\theta \quad (|\theta| \leq 1).$$

2324. $\int_0^1 \frac{x^9}{\sqrt{1+x}} dx.$

解 由于 $\frac{x^9}{\sqrt{2}} \leq \frac{x^9}{\sqrt{1+x}} \leq x^9$ ($0 \leq x \leq 1$), 从而,

$$\frac{1}{\sqrt{2}} \int_0^1 x^9 dx \leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx$$

$$\leq \int_0^1 x^9 dx,$$

即

$$\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \frac{1}{10}.$$

2325. $\int_0^{100} \frac{e^{-x}}{x+100} dx.$

$$\begin{aligned}
 \text{解 } I &= \int_0^{50} \frac{e^{-x}}{x+100} dx + \int_{50}^{100} \frac{e^{-x}}{x+100} dx \\
 &= \frac{1}{100+\xi_1} \int_0^{50} e^{-x} dx + \frac{1}{100+\xi_2} \int_{50}^{100} e^{-x} dx \\
 &= \frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_2}, \text{ 其中 } 0 \leq \xi_1 \leq 50, \\
 &\quad 50 \leq \xi_2 \leq 100.
 \end{aligned}$$

显然

$$\begin{aligned}
 &\frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_2} \\
 &\leq \frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_1} \\
 &= \frac{1-e^{-100}}{100+\xi_1} < \frac{1}{100}, \\
 &\frac{1-e^{-50}}{100+\xi_1} + \frac{e^{-50}-e^{-100}}{100+\xi_2} \\
 &> \frac{1-e^{-50}}{100+\xi_1} \geq \frac{1-e^{-50}}{150} > \frac{1}{200},
 \end{aligned}$$

故 $\frac{1}{200} < I < \frac{1}{100}$, 即 $I = 0.01 - 0.005\theta$, $0 < \theta < 1$.

另外, 按中值定理, 可写

$$\begin{aligned}
 I &= \int_0^{100} \frac{e^{-x}}{x+100} dx = \frac{1}{\xi+100} \int_0^{100} e^{-x} dx \\
 &= \frac{1}{\xi+100} \left(1 - \frac{1}{e^{100}} \right),
 \end{aligned}$$

其中 $0 \leq \xi \leq 100$, 如果改写 I 为

$$I = 0.01 - 0.005\theta,$$

则有

$$\theta = f(\xi) = \frac{2}{100 + \xi} \left(\xi + \frac{100}{e^{100}} \right).$$

易见导数

$$f'(\xi) = \frac{200(1 - e^{-100})}{(100 + \xi)^2} > 0,$$

$f(\xi)$ 单调上升, 故在 $[0, 100]$ 上有 $f(0) \leq f(\xi) \leq f(100)$, 也即有

$$\frac{2}{e^{100}} \leq \theta \leq 1 + \frac{1}{e^{100}}.$$

根据前面的估计 $0 < \theta < 1$, 综合起来, 便有

$$\frac{2}{e^{100}} \leq \theta < 1.$$

这个结果比原来的估计又好了一些. 如果更精确一些, 采用些近似计算方法, 还可进一步明确 θ 的数值范围. 此处从略.

2326. 证明等式

$$(a) \lim_{n \rightarrow \infty} \int_0^1 \frac{x^n}{1+x} dx = 0; \quad (b) \lim_{n \rightarrow \infty} \int_0^{\frac{\pi}{2}} \sin^n x dx = 0.$$

$$\begin{aligned} \text{证 (a)} \quad \lim_{n \rightarrow \infty} \int_0^1 \frac{x^n}{1+x} dx &= \lim_{n \rightarrow \infty} \frac{1}{1+\xi_n} \int_0^1 x^n dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{1+\xi_n} \cdot \frac{1}{n+1} = 0; \end{aligned}$$

(6) 任意给定 $\varepsilon > 0$, 且设 $\varepsilon < \frac{\pi}{2}$, 则

$$\begin{aligned} 0 &\leq \int_0^{\frac{\pi}{2}} \sin^n x dx \leq \int_0^{\frac{\pi}{2}-\varepsilon} \sin^n x dx + \varepsilon \\ &\leq \varepsilon + \left(\frac{\pi}{2} - \varepsilon\right) \sin^n \left(\frac{\pi}{2} - \varepsilon\right). \end{aligned}$$

当 $n \rightarrow \infty$ 时, 上述不等式的第二项趋于零, 于是

$$\lim_{n \rightarrow \infty} \int_0^{\frac{\pi}{2}} \sin^n x dx = 0.$$

2327. 设函数 $f(x)$ 在 (a, b) 上连续, 而 $\varphi(x)$ 在 (a, b) 上连续且在 (a, b) 上可微分, 并且

$$\varphi'(x) \geq 0 \quad \text{当} \quad a < x < b.$$

应用部分积分法及第一中值定理以证明第二中值定理.

证 设 $F(x) = \int_a^x f(t) dt$, 则

$$\begin{aligned} \int_a^b f(x)\varphi(x) dx &= \int_a^b \varphi(x) dF(x) \\ &= F(x)\varphi(x) \Big|_a^b - \int_a^b F(x)\varphi'(x) dx \\ &= F(b)\varphi(b) - F(a)\varphi(a) - F(\xi) \int_a^b \varphi'(x) dx \\ &= F(b)\varphi(b) - F(a)\varphi(a) - F(\xi)[\varphi(b) - \varphi(a)]^*) \\ &= \varphi(b)[F(b) - F(\xi)] + \varphi(a)[F(\xi) - F(a)] \\ &= \varphi(b) \int_{\xi}^b f(x) dx + \varphi(a) \int_a^{\xi} f(x) dx. \end{aligned}$$

*) 一般数学分析中已有第二中值定理的证明, 本题限用部分积分法证明, 应加 $\varphi'(x)$ 在 (a, b) 上连续的条件.

利用第二中值定理, 估计积分:

$$2328. \int_{100\pi}^{200\pi} \frac{\sin x}{x} dx.$$

解 设 $f(x) = \sin x$, $\varphi(x) = \frac{1}{x}$, 则 $f(x)$ 及 $\varphi(x)$ 在 $[100\pi, 200\pi]$ 上满足第二中值定理的条件, 又 $\varphi(x) = \frac{1}{x}$ 单调下降且不为负, 于是,

$$\begin{aligned} \int_{100\pi}^{200\pi} \frac{\sin x}{x} dx &= \frac{1}{100\pi} \int_{100\pi}^{\xi} \sin x dx \\ &= \frac{1 - \cos \xi}{100\pi} = \frac{\sin^2 \frac{\xi}{2}}{50\pi} = \frac{\theta}{50\pi}, \end{aligned}$$

其中 $100\pi \leq \xi \leq 200\pi$ 及 $0 \leq \theta \leq 1$.

$$2329. \int_a^b \frac{e^{-ax}}{x} \sin x dx \quad (a \geq 0; 0 < a < b).$$

解 设 $f(x) = \sin x$, $\varphi(x) = \frac{e^{-ax}}{x}$, 同上题, 有

$$\begin{aligned} \int_a^b \frac{e^{-ax}}{x} \sin x dx &= \frac{e^{-a\xi}}{a} \int_a^{\xi} \sin x dx \\ &= \frac{1}{ae^{a\xi}} (\cos a - \cos \xi) \\ &= \frac{2}{a} e^{-a\xi} \sin \frac{a+\xi}{2} \sin \frac{a-\xi}{2} = \frac{2}{a} \theta, \end{aligned}$$

其中 $a \leq \xi \leq b$ 及 $|\theta| < 1$.

2330. $\int_a^b \sin x^2 dx \quad (0 < a < b)$.

解 设 $x = \sqrt{t}$. 则

$$\int_a^b \sin x^2 dx = \frac{1}{2} \int_{a^2}^{b^2} \frac{\sin t}{\sqrt{t}} dt .$$

其次, 设 $f(t) = \sin t$, $\varphi(t) = (\sqrt{t})^{-1}$, 则 $\varphi(t)$ 单调下降, 且 $\varphi(t) > 0$, 于是

$$\begin{aligned} \frac{1}{2} \int_{a^2}^{b^2} \frac{\sin t}{\sqrt{t}} dt &= \frac{1}{2a} \int_{a^2}^{\xi} \sin t dt \\ &= \frac{1}{2a} (\cos a^2 - \cos \xi) = \frac{1}{a} \sin \frac{\xi + a^2}{2} \sin \frac{\xi - a^2}{2} \\ &= \frac{1}{a} \theta, \end{aligned}$$

其中 $a^2 \leq \xi \leq b^2$, $|\theta| \leq 1$. 所以

$$\int_a^b \sin x^2 dx = \frac{\theta}{a} \quad (|\theta| \leq 1) .$$

2331. 设函数 $\varphi(x)$ 及 $\psi(x)$ 和它们的平方在区间 (a, b) 上可积分. 证明哥西—布尼雅可夫斯基不等式

$$\left\{ \int_a^b \varphi(x) \psi(x) dx \right\}^2 \leq \int_a^b \varphi^2(x) dx \int_a^b \psi^2(x) dx .$$

证 证法一: 我们有

$$\left(\int_a^b \varphi^2(x) dx \right) \cdot \left(\int_a^b \psi^2(x) dx \right)$$

$$\begin{aligned}
& - \left(\int_a^b \varphi(x) \psi(x) dx \right)^2 \\
&= \frac{1}{2} \left(\int_a^b \varphi^2(x) dx \right) \cdot \left(\int_a^b \psi^2(y) dy \right) \\
& \quad + \frac{1}{2} \left(\int_a^b \psi^2(x) dx \right) \cdot \left(\int_a^b \varphi^2(y) dy \right) \\
& \quad - \left(\int_a^b \varphi(x) \psi(x) dx \right) \cdot \left(\int_a^b \varphi(y) \psi(y) dy \right) \\
&= \frac{1}{2} \int_a^b \left\{ \int_a^b [\varphi(x)\psi(y) - \psi(x)\varphi(y)]^2 dx \right\} dy \\
&\geq 0,
\end{aligned}$$

故

$$\left\{ \int_a^b \varphi(x) \psi(x) dx \right\}^2 \leq \int_a^b \varphi^2(x) dx \cdot \int_a^b \psi^2(x) dx.$$

证法二：考虑积分

$$\int_a^b [\varphi(x) - \lambda \psi(x)]^2 dx,$$

其中 λ 为任意实数。从而

$$\begin{aligned}
& \int_a^b \varphi^2(x) dx - 2\lambda \int_a^b \varphi(x) \psi(x) dx \\
& \quad + \lambda^2 \int_a^b \psi^2(x) dx \geq 0.
\end{aligned}$$

这是关于变数 λ 的不等式，左端是二次三项式，于是其判别式

$$\left\{ \int_a^b \varphi(x) \psi(x) dx \right\}^2 - \int_a^b \varphi^2(x) dx$$

$$\int_a^b \psi^2(x) dx \leq 0,$$

即

$$\left\{ \int_a^b \varphi(x)\psi(x) dx \right\}^2 \\ \leq \int_a^b \varphi^2(x) dx \cdot \int_a^b \psi^2(x) dx.$$

2332. 设函数 $f(x)$ 在闭区间 $[a, b]$ 上连续可微分且 $f(a) = 0$, 证明不等式

$$M^2 \leq (b-a) \int_a^b f'^2(x) dx,$$

其中 $M = \sup_{a \leq x \leq b} |f(x)|$.

证 设 x 为 $[a, b]$ 上任一点, 则利用哥西—布尼雅可夫斯基不等式得到

$$\left\{ \int_a^x f'(x) dx \right\}^2 \leq \int_a^x 1 \cdot dx \cdot \int_a^x f'^2(x) dx,$$

即

$$f^2(x) = [f(x) - f(a)]^2 \leq (x-a) \int_a^x f'^2(x) dx \\ \leq (b-a) \int_a^b f'^2(x) dx.$$

由此可知

$$M^2 = \sup_{x \in [a, b]} f^2(x) \leq (b-a) \int_a^b f'^2(x) dx.$$

2333. 证明等式:

$$\lim_{n \rightarrow \infty} \int_n^{n+p} \frac{\sin x}{x} dx = 0.$$

证 证法一：应用第一中值定理，知

$$\lim_{n \rightarrow \infty} \int_n^{n+p} \frac{\sin x}{x} dx = \lim_{n \rightarrow \infty} \frac{\sin \xi_n}{\xi_n} \cdot p = 0,$$

其中 ξ_n 为界于 n 与 $n+p$ 之间的某值。

证法二：应用第二中值定理，得

$$\begin{aligned} \left| \int_n^{n+p} \frac{\sin x}{x} dx \right| &= \frac{1}{n} \left| \int_n^{\xi'_n} \sin x dx \right| \\ &= \frac{1}{n} \left| \cos n - \cos \xi'_n \right| \leq \frac{2}{n} \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

其中 ξ'_n 是界于 n 与 $n+p$ 之间的某值。于是

$$\lim_{n \rightarrow \infty} \int_n^{n+p} \frac{\sin x}{x} dx = 0.$$

§4. 广义积分

1° 函数的广义可积分性 若函数 $f(x)$ 于每一个有穷区间 $[a, b)$ 上依寻常的意义是可积分的，则可定义

$$\int_a^{+\infty} f(x) dx = \lim_{b \rightarrow +\infty} \int_a^b f(x) dx. \quad (1)$$

若函数 $f(x)$ 于点 b 的邻域内无界且于每一个区间 $(a, b-\varepsilon)$ ($\varepsilon > 0$) 内依寻常的意义是可积分的，则取

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{b-\varepsilon} f(x) dx. \quad (2)$$

若极限(1)或(2)存在, 则对应的积分称为收敛的, 在相反的情形则称为发散的.

2° 哥西准则 积分(1)收敛的充要条件为对于任意的 $\varepsilon > 0$, 存在有数 $b = b(\varepsilon)$, 当 $b' > b$ 及 $b'' > b$ 时, 下面的不等式成立

$$\left| \int_{b'}^{b''} f(x) dx \right| < \varepsilon.$$

同样地对形状为(2)的积分可述出哥西准则.

3° 绝对收敛的判别法 若 $|f(x)|$ 是广义可积分的, 则函数 $f(x)$ 的对应的积分(1)或(2)称为绝对收敛的, 而且显然也是收敛的积分.

比较判别法 I. 设当 $x \geq a$ 时 $|f(x)| \leq F(x)$.

若 $\int_a^{+\infty} F(x) dx$ 收敛, 则积分 $\int_a^{+\infty} f(x) dx$ 绝对收敛.

比较判别法 II. 若 $\varphi(x) > 0$ 及当 $x \rightarrow +\infty$ 时,

$$\varphi(x) = O^*(\psi(x)),$$

则积分 $\int_a^{+\infty} \varphi(x) dx$ 及 $\int_a^{+\infty} \psi(x) dx$ 同时收敛或同时发散.

就特别情形来说, 若当 $x \rightarrow +\infty$ 时, $\varphi(x) \sim \psi(x)$, 则上面的结果也成立.

比较判别法 III. (a) 设当 $x \rightarrow +\infty$ 时,

$$f(x) = O^*\left(\frac{1}{x^p}\right).$$

在这种情况下, 当 $p > 1$ 时, 积分(1)收敛; 当 $p \leq 1$ 时, 积分(1)发散.

(6) 设当 $x \rightarrow b-0$ 时,

$$f(x) = O^*\left[\frac{1}{(b-x)^p}\right].$$

在这种情况下, 当 $p < 1$ 时, 积分 (2) 收敛; 当 $p \geq 1$ 时, 积分 (2) 发散.

4° 收敛性的较精密的判别法 若 (1) 当 $x \rightarrow +\infty$ 时, 函数 $\varphi(x)$ 单调地趋近于零; (2) 函数 $f(x)$ 有有界的原函数

$$F(x) = \int_a^x f(\xi) d\xi,$$

则积分

$$\int_a^{+\infty} f(x)\varphi(x) dx$$

收敛, 但一般地说, 并非绝对收敛.

特殊情形, 若 $p > 0$, 则积分

$$\int_a^{+\infty} \frac{\cos x}{x^p} dx \text{ 及 } \int_a^{+\infty} \frac{\sin x}{x^p} dx \quad (a > 0)$$

收敛.

5° 在哥西意义上的主值 若函数 $f(x)$ 对任意的 $\varepsilon > 0$ 积分

$$\int_a^{c-\varepsilon} f(x) dx \text{ 及 } \int_{c+\varepsilon}^b f(x) dx \quad (a < c < b)$$

存在, 则在哥西意义上的主值 ($V \cdot P \cdot$) 为

$$V \cdot P \cdot \int_a^b f(x) dx$$

$$= \lim_{\varepsilon \rightarrow +0} \left[\int_a^{c-\varepsilon} f(x) dx + \int_{c+\varepsilon}^b f(x) dx \right].$$

相仿地, $V.P. \int_{-\infty}^{+\infty} f(x) dx = \lim_{a \rightarrow +\infty} \int_{-a}^a f(x) dx.$

计算下列积分:

2334. $\int_a^{+\infty} \frac{dx}{x^2} \quad (a > 0).$

解 由于

$$\lim_{b \rightarrow +\infty} \int_a^b \frac{dx}{x^2} = \lim_{b \rightarrow +\infty} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{a},$$

所以

$$\int_a^{+\infty} \frac{dx}{x^2} = \frac{1}{a}.$$

2335. $\int_0^1 \ln x dx.$

解 由于

$$\lim_{\varepsilon \rightarrow +0} \int_{\varepsilon}^1 \ln x dx = \lim_{\varepsilon \rightarrow +0} (\varepsilon - \varepsilon \ln \varepsilon - 1) = -1,$$

所以

$$\int_0^1 \ln x dx = -1.$$

2336. $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}.$

解 由于

$$\lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2} + \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{1+x^2} = \pi,$$

所以

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi.$$

$$2337. \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}.$$

解 由于

$$\begin{aligned} & \lim_{\varepsilon \rightarrow +0} \int_{-1+\varepsilon}^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{\varepsilon' \rightarrow +0} \int_0^{1-\varepsilon'} \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{\varepsilon \rightarrow +0} [-\arcsin(-1+\varepsilon)] + \lim_{\varepsilon' \rightarrow +0} \arcsin(1-\varepsilon') \\ &= \pi, \end{aligned}$$

所以

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi.$$

$$2338. \int_2^{+\infty} \frac{dx}{x^2+x-2}.$$

解 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_2^b \frac{dx}{x^2+x-2} = \lim_{b \rightarrow +\infty} \left(\frac{1}{3} \ln \frac{x-1}{x+2} \right) \Big|_2^b \\ &= \frac{1}{3} \lim_{b \rightarrow +\infty} \left(\ln \frac{b-1}{b+2} + 2 \ln 2 \right) = \frac{2}{3} \ln 2, \end{aligned}$$

所以

$$\int_2^{+\infty} \frac{dx}{x^2+x-2} = \frac{2}{3} \ln 2.$$

$$2339. \int_{-\infty}^{+\infty} \frac{dx}{(x^2+x+1)^2}.$$

$$\text{解 } \int \frac{dx}{(x^2+x+1)^2} = \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}} \operatorname{arc\,tg} \frac{2x+1}{\sqrt{3}} + C.$$

由于

$$\begin{aligned} & \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{(x^2+x+1)^2} + \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{(x^2+x+1)^2} \\ &= \lim_{a \rightarrow -\infty} \left\{ \left(\frac{1}{3} + \frac{4}{3\sqrt{3}} \operatorname{arc\,tg} \frac{1}{\sqrt{3}} \right) \right. \\ & \quad \left. - \left[\frac{2a+1}{3(a^2+a+1)} + \frac{1}{3\sqrt{3}} \operatorname{arc\,tg} \frac{2a+1}{\sqrt{3}} \right] \right\} \\ & \quad + \lim_{b \rightarrow +\infty} \left\{ \left[\frac{2b+1}{3(b^2+b+1)} + \frac{4}{3\sqrt{3}} \operatorname{arc\,tg} \frac{2b+1}{\sqrt{3}} \right] \right. \\ & \quad \left. - \left(\frac{1}{3} + \frac{4}{3\sqrt{3}} \operatorname{arc\,tg} \frac{1}{\sqrt{3}} \right) \right\} \\ &= \frac{4\pi}{3\sqrt{3}}, \end{aligned}$$

所以

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+x+1)^2} = \frac{4\pi}{3\sqrt{3}}.$$

*) 利用1921题的递推公式.

$$2340. \int_0^{+\infty} \frac{dx}{1+x^3}.$$

解 由于

$$\begin{aligned}
& \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{1+x^3} \\
&= \lim_{b \rightarrow +\infty} \left[\frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2x-1}{\sqrt{3}} \right] \Big|_0^b \\
&= \frac{2\pi}{3\sqrt{3}},
\end{aligned}$$

所以

$$\int_0^{+\infty} \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}.$$

*) 利用1881题的结果。

2341. $\int_0^{+\infty} \frac{x^2+1}{x^4+1} dx.$

解 由于

$$\begin{aligned}
& \lim_{\substack{b \rightarrow +\infty \\ a \rightarrow +0}} \int_a^b \frac{x^2+1}{x^4+1} dx \\
&= \lim_{\substack{b \rightarrow +\infty \\ a \rightarrow +0}} \left(\frac{1}{\sqrt{2}} \operatorname{arc} \operatorname{tg} \frac{x^2-1}{x\sqrt{2}} \right) \Big|_a^b = \frac{\pi}{\sqrt{2}},
\end{aligned}$$

所以

$$\int_0^{+\infty} \frac{x^2+1}{x^4+1} dx = \frac{\pi}{\sqrt{2}}.$$

*) 利用1712题的结果。

2342. $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}.$

解 先求 $\int \frac{dx}{(2-x)\sqrt{1-x}}$. 设 $\sqrt{1-x}=t$, 则

$$x=1-t^2, dx=-2tdt, 2-x=1+t^2.$$

代入得

$$\int \frac{dx}{(2-x)\sqrt{1-x}} = -2 \int \frac{dt}{1+t^2}$$

$$= -2 \operatorname{arc} \operatorname{tg} t + C = -2 \operatorname{arc} \operatorname{tg} \sqrt{1-x} + C.$$

由于

$$\begin{aligned} & \lim_{\varepsilon \rightarrow +0} \int_0^{1-\varepsilon} \frac{dx}{(2-x)\sqrt{1-x}} \\ &= \lim_{\varepsilon \rightarrow +0} \left(-2 \operatorname{arc} \operatorname{tg} \sqrt{1-x} \Big|_0^{1-\varepsilon} \right) \\ &= -2 \lim_{\varepsilon \rightarrow +0} \left[\operatorname{arc} \operatorname{tg} \sqrt{1-(1-\varepsilon)} - \frac{\pi}{4} \right] \\ &= \frac{\pi}{2}, \end{aligned}$$

所以

$$\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}} = \frac{\pi}{2}.$$

2343. $\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^5+x^{10}}}.$

解 设 $\sqrt{1+x^5+x^{10}}=t-x^5$. 则当 $1 \leq x < +\infty$ 时,

$1 + \sqrt{3} \leq t < +\infty$. 代入得

$$\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^5+x^{10}}}$$

$$= \frac{2}{5} \int_{1+\sqrt{3}}^{+\infty} \frac{dt}{t^2-1} = \frac{1}{5} \ln \frac{t-1}{t+1} \Big|_{1+\sqrt{3}}^{+\infty} \quad *)$$

$$= \frac{1}{5} \ln 1 - \frac{1}{5} \ln \frac{\sqrt{3}}{2+\sqrt{3}} = \frac{1}{5} \ln \left(1 + \frac{2}{\sqrt{3}} \right).$$

*) 牛顿—莱不尼兹公式对于广义积分也成立。例如

$$\int_a^{+\infty} f(x) dx = F(+\infty) - F(a) = F(x) \Big|_a^{+\infty},$$

其中 $F(+\infty)$ 是一个符号，代表 $\lim_{x \rightarrow +\infty} F(x)$ (假定此极限存在有限)，下同，不再说明。

2344. $\int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx.$

解 我们有

$$\begin{aligned} \int \frac{x \ln x}{(1+x^2)^2} dx &= -\frac{1}{2} \int \ln x d \left(\frac{1}{1+x^2} \right) \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{x(1+x^2)} \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2} \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx \\ &= -\frac{\ln x}{2(1+x^2)} + \frac{1}{4} \ln \frac{x^2}{1+x^2} + C. \end{aligned}$$

由于

$$\lim_{\delta \rightarrow +0} \int_{\delta}^{\delta} \frac{x \ln x}{(1+x^2)^2} dx$$

$$\begin{aligned}
&= \lim_{\substack{\varepsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left\{ -\frac{\ln x}{2(1+x^2)} + \frac{1}{4} \ln \frac{x^2}{1+x^2} \right\} \Big|_b \\
&= \lim_{\substack{\varepsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left\{ -\frac{\ln b}{2(1+b^2)} + \frac{\ln \varepsilon}{2(1+\varepsilon^2)} + \frac{1}{4} \ln \frac{b^2}{b^2+1} \right. \\
&\quad \left. - \frac{1}{4} \ln \frac{\varepsilon^2}{\varepsilon^2+1} \right\} \\
&= \lim_{\varepsilon \rightarrow +0} \left\{ -\frac{\varepsilon^2}{2(\varepsilon^2+1)} \ln \varepsilon + \frac{1}{4} \ln(\varepsilon^2+1) \right\} \\
&= 0,
\end{aligned}$$

所以

$$\int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx = 0.$$

注 $\varepsilon \rightarrow +0$ 与 $b \rightarrow +\infty$ 的极限过程是独立的，因此可分别取极限。

2345. $\int_0^{+\infty} \frac{\operatorname{arc} \operatorname{tg} x}{(1+x^2)^{\frac{3}{2}}} dx.$

解 设 $x = \operatorname{tg} t$, 则

$$\begin{aligned}
\int_0^{+\infty} \frac{\operatorname{arc} \operatorname{tg} x}{(1+x^2)^{\frac{3}{2}}} dx &= \int_0^{\frac{\pi}{2}} \frac{t \sec^2 t dt}{\sec^3 t} \\
&= (t \sin t + \cos t) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1.
\end{aligned}$$

2346. $\int_0^{+\infty} e^{-ax} \cos bx dx \quad (a > 0).$

$$\begin{aligned}
 \text{解} \quad & \int_0^{+\infty} e^{-ax} \cos bx dx \\
 &= \left(\frac{-a \cos bx + b \sin bx}{a^2 + b^2} e^{-ax} \right)^* \Big|_0^{+\infty} \\
 &= \frac{a}{a^2 + b^2}.
 \end{aligned}$$

*) 利用1828题的结果.

$$2347. \int_0^{+\infty} e^{-ax} \sin bx dx \quad (a > 0).$$

$$\begin{aligned}
 \text{解} \quad & \int_0^{+\infty} e^{-ax} \sin bx dx \\
 &= \left(\frac{-a \sin bx - b \cos bx}{a^2 + b^2} e^{-ax} \right)^* \Big|_0^{+\infty} \\
 &= \frac{b}{a^2 + b^2}.
 \end{aligned}$$

*) 利用1829题的结果.

利用递推公式计算下列广义积分(n 为自然数):

$$2348. I_n = \int_0^{+\infty} x^n e^{-x} dx.$$

$$\begin{aligned}
 \text{解} \quad I_n &= \int_0^{+\infty} x^n d(-e^{-x}) \\
 &= -x^n e^{-x} \Big|_0^{+\infty} + n \int_0^{+\infty} x^{n-1} e^{-x} dx \\
 &= n \int_0^{+\infty} x^{n-1} e^{-x} dx = n I_{n-1},
 \end{aligned}$$

即 $I_n = nI_{n-1}$. 利用此递推公式及

$$I_0 = \int_0^{+\infty} e^{-x} dx = 1$$

容易得到

$$I_n = n(n-1)\cdots 2 \cdot 1 I_0 = n!$$

$$2349^+. \quad I_n = \int_{-\infty}^{+\infty} \frac{dx}{(ax^2 + 2bx + c)^n} \quad (ac - b^2 > 0).$$

$$\begin{aligned} \text{解} \quad I_n &= \frac{ax+b}{2(n-1)(ac-b^2)(ax^2+2bx+c)^{n-1}} \Big|_{-\infty}^{+\infty} \\ &\quad + \frac{2n-3}{n-1} \cdot \frac{a}{2(ac-b^2)} I_{n-1} \quad *) \\ &= \frac{2n-3}{2(n-1)} \cdot \frac{a}{ac-b^2} I_{n-1}, \end{aligned}$$

即

$$I_n = \frac{2n-3}{2(n-1)} \cdot \frac{a}{ac-b^2} I_{n-1} \quad (n \geq 1).$$

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \frac{dx}{ax^2 + 2bx + c} \\ &= \frac{\operatorname{sgn} a}{\sqrt{ac-b^2}} \operatorname{arc} \operatorname{tg} \frac{|a| \left(x + \frac{b}{a} \right)}{\sqrt{ac-b^2}} \Big|_{-\infty}^{+\infty} \\ &= \frac{\pi \operatorname{sgn} a}{\sqrt{ac-b^2}}. \end{aligned}$$

利用递推公式及 I_1 容易得到

$$I_n = \frac{(2n-3)(2n-5)\cdots 3 \cdot 1}{(2n-2)(2n-4)\cdots 4 \cdot 2} \cdot \frac{\pi a^{n-1} \operatorname{sgn} a}{(ac-b^2)^{n-\frac{1}{2}}}$$

$$= \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{\pi a^{n-1} \operatorname{sgn} a}{(ac-b^2)^{n-\frac{1}{2}}}$$

*) 利用1921题的结果。

$$2350^+ \quad I_n = \int_1^{+\infty} \frac{dx}{x(x+1)\cdots(x+n)}.$$

解 由于 $x^{n+1} \cdot \frac{1}{x(x+1)\cdots(x+n)} \rightarrow 1$ (当 $x \rightarrow +\infty$ 时), 且 $n+1 > 1$, 所以积分 I_n 收敛。

其次, 我们来计算 I_n . 由于

$$\begin{aligned} & \frac{1}{x(x+1)\cdots(x+n)} \\ &= \frac{1}{n!x} - \frac{1}{(n-1)!(x+1)} + \frac{1}{2!(n-2)!(x+2)} \\ & \quad - \cdots + (-1)^k \frac{1}{k!(n-k)!(x+k)} \\ & \quad + \cdots + (-1)^n \frac{1}{n!(x+n)}, \end{aligned}$$

所以

$$\begin{aligned} I_n &= \frac{1}{n!} \int_1^{+\infty} \sum_{k=0}^n C_n^k (-1)^k \frac{dx}{x+k} \\ &= \frac{1}{n!} \sum_{k=0}^n (-1)^k C_n^k \ln(x+k) \Big|_1^{+\infty}, \end{aligned}$$

其中 C_n^k 为从 n 个元素中每次取 k 个的组合数。

对于 n , 不论是偶数还是奇数, 用上限代入 (此处理解为趋近于无穷时的极限) 后均为零。事实上,

当 $n = 2m$ 时,

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^k C_{2n}^k \ln(x+k) \\ &= \ln \frac{x \cdot (x+2)^{C_{2n}^2} \cdots (x+2m)^{C_{2n}^{2m}}}{(x+1)^{C_{2n}^1} (x+3)^{C_{2n}^3} \cdots (x+2m-1)^{C_{2n}^{2m-1}}} \end{aligned}$$

由于

$$1 + C_{2m}^2 + \cdots + C_{2m}^{2m} = C_{2m}^1 + C_{2m}^3 + \cdots + C_{2m}^{2m-1},$$

所以, 当 $m \rightarrow +\infty$ 时

$$\sum_{k=0}^{2n} (-1)^k C_{2n}^k \ln(x+k) \rightarrow \ln 1 = 0;$$

当 $n = 2m - 1$ 时,

$$\begin{aligned} & \sum_{k=0}^{2m-1} (-1)^k C_{2m-1}^k \ln(x+k) \\ &= \ln \frac{x(x+2)^{C_{2m-1}^2} \cdots (x+2m-2)^{C_{2m-1}^{2m-2}}}{(x+1)^{C_{2m-1}^1} (x+3)^{C_{2m-1}^3} \cdots (x+2m-2)^{C_{2m-1}^{2m-1}}} \\ & \rightarrow 0 \quad (\text{当 } m \rightarrow +\infty \text{ 时}). \end{aligned}$$

最后我们得到

$$I_n = \frac{1}{n!} \sum_{k=0}^n (-1)^{k+1} C_n^k \ln(1+k).$$

2351. $I_2 = \int_0^1 \frac{x^2 dx}{\sqrt{(1-x)(1+x)}}.$

解 由于 $\sqrt{\frac{x^2}{1-x}} \cdot \frac{1}{\sqrt{(1-x)(1+x)}} \rightarrow \frac{1}{2}$

(当 $x \rightarrow 1-0$ 时),

且 $p = \frac{1}{2} < 1$, 所以积分 I_n 收敛.

其次, 设 $x = \sin t$, 则

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t \, dt = \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & \text{当 } n=2k \text{ 时;}^*) \\ \frac{(2k-2)!!}{(2k-1)!!}, & \text{当 } n=2k-1 \text{ 时.} \end{cases}$$

*) 利用2281题的结果.

2352. $I_n = \int_0^{+\infty} \frac{dx}{\operatorname{ch}^{n+1} x}$.

解 设 $x = \ln \left(\operatorname{tg} \frac{t}{2} \right)$, 则

$$\text{当 } 0 \leq x < +\infty \text{ 时, } \frac{\pi}{2} \leq t \leq \pi,$$

$$I_n = \int_0^{+\infty} \frac{dx}{\operatorname{ch}^{n+1} x} = \int_{\frac{\pi}{2}}^{\pi} \sin^n t \, dt = \int_0^{\frac{\pi}{2}} \cos^n u \, du$$

$$= \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & \text{当 } n=2k \text{ 时;}^*) \\ \frac{(2k-2)!!}{(2k-1)!!}, & \text{当 } n=2k-1 \text{ 时.} \end{cases}$$

*) 利用2282题的结果.

2353. (a) $\int_0^{\frac{\pi}{2}} \ln \sin x \, dx$; (b) $\int_0^{\frac{\pi}{2}} \ln \cos x \, dx$.

解 先证明它们是收敛的. 事实上, 当 $x \rightarrow +0$ 时, $\sqrt{x} \cdot \ln \sin x \rightarrow 0$, 所以, 积分

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, dx$$

收敛。

同法可证积分

$$\int_0^{\frac{\pi}{2}} \ln \cos x \, dx$$

也收敛。

其次，求这两个积分的值。设 $t = \frac{\pi}{2} - x$ ，则有

$$\int_0^{\frac{\pi}{2}} \ln \cos x \, dx = \int_0^{\frac{\pi}{2}} \ln \sin x \, dx = A.$$

相加得

$$\begin{aligned} 2A &= \int_0^{\frac{\pi}{2}} (\ln \sin x + \ln \cos x) \, dx \\ &= \int_0^{\frac{\pi}{2}} \ln \left(\frac{1}{2} \sin 2x \right) \, dx \\ &= \int_0^{\frac{\pi}{2}} \ln \sin 2x \, dx - \ln 2 \cdot \int_0^{\frac{\pi}{2}} dx \\ &= \frac{1}{2} \int_0^{\pi} \ln \sin t \, dt - \frac{\pi}{2} \ln 2 \\ &= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \ln \sin t \, dt + \int_{\frac{\pi}{2}}^{\pi} \ln \sin t \, dt \right) - \frac{\pi}{2} \ln 2 \\ &= \int_0^{\frac{\pi}{2}} \ln \sin t \, dt - \frac{\pi}{2} \ln 2 \\ &= A - \frac{\pi}{2} \ln 2, \end{aligned}$$

于是, $2A = A - \frac{\pi}{2} \ln 2$, $A = -\frac{\pi}{2} \ln 2$, 即

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, dx = \int_0^{\frac{\pi}{2}} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2.$$

2354. 求:

$$\int_E e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} \, dx,$$

其中 E 表区间 $(0, +\infty)$ 中使被积分式有意义的一切 x 值所成之集合.

解 $\int_E e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} \, dx$

$$= \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} \, dx, *$$

对于广义积分 $\int_{2k\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} \, dx$

作如下处理:

$$\begin{aligned} & \int_{2k\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} \, dx \\ &= \int_{2k\pi}^{(2k+\frac{1}{2})\pi} e^{-\frac{x}{2}} \frac{\cos x - \sin x}{\sqrt{\sin x}} \, dx + \\ & \quad + \int_{(2k+\frac{1}{2})\pi}^{(2k+1)\pi} e^{-\frac{x}{2}} \frac{\sin x - \cos x}{\sqrt{\sin x}} \, dx \end{aligned}$$

*记号 $\sum_{k=0}^{\infty} S_k$ 理解为极限 $\lim_{n \rightarrow +\infty} \sum_{k=0}^n S_k$, 以后题解中不再说明.

$$\begin{aligned}
 &= 2e^{-\frac{x}{2}} \sqrt{\sin x} \Big|_{\frac{x}{2}}^{(2k+\frac{1}{2})x} - 2e^{-\frac{x}{2}} \sqrt{\sin x} \Big|_{\frac{x}{2}}^{(2k+1)x} \\
 &= 2\sqrt[4]{8} \cdot e^{-kx} \cdot e^{-\frac{x}{2}}.
 \end{aligned}$$

由于

$$\sum_{i=0}^n 2\sqrt[4]{8} \cdot e^{-ix} e^{-\frac{x}{2}} = 2\sqrt[4]{8} \cdot e^{-\frac{x}{2}} \cdot \frac{1 - e^{-(n+1)x}}{1 - e^{-x}}.$$

当 $n \rightarrow +\infty$ 时, 上式的极限为 $2\sqrt[4]{8} \cdot e^{-\frac{x}{2}} \cdot \frac{1}{1 - e^{-x}}$.

于是,

$$\int_0^{\frac{\pi}{2}} e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx = \frac{2\sqrt[4]{8} e^{-\frac{\pi}{4}}}{1 - e^{-\frac{\pi}{2}}}.$$

2355. 证明等式

$$\int_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx = \frac{1}{a} \int_0^{+\infty} f(\sqrt{x^2 + 4ab}) dx,$$

其中 $a > 0$, $b > 0$ (假定等式左端的积分有意义).

证 设 $ax - \frac{b}{x} = t$, 则

当 $0 < x < +\infty$ 时, $-\infty < t < +\infty$,

$$ax + \frac{b}{x} = \sqrt{t^2 + 4ab}.$$

将此二式相加得

$$x = \frac{1}{2a} (t + \sqrt{t^2 + 4ab}).$$

从而有

$$dx = \frac{1}{2a} \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt.$$

代入欲证的等式左端，得

$$\begin{aligned} & \int_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx \\ &= \frac{1}{2a} \int_{-\infty}^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\ &= \frac{1}{2a} \int_{-\infty}^0 f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\ & \quad + \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\ &= \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{\sqrt{t^2 + 4ab} - t}{\sqrt{t^2 + 4ab}} dt \\ & \quad + \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \cdot \frac{t + \sqrt{t^2 + 4ab}}{\sqrt{t^2 + 4ab}} dt \\ &= \frac{1}{2a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) \\ & \quad \cdot \frac{\sqrt{t^2 + 4ab} - t + \sqrt{t^2 + 4ab} + t}{\sqrt{t^2 + 4ab}} dt \\ &= \frac{1}{a} \int_0^{+\infty} f(\sqrt{t^2 + 4ab}) dt, \end{aligned}$$

于是

$$\int_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx = \frac{1}{a} \int_0^{+\infty} f(\sqrt{x^2 + 4ab}) dx.$$

2356. 数

$$M[f] = \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(\xi) d\xi$$

称为函数 $f(x)$ 在区间 $(0, +\infty)$ 上的平均值, 求下列函数的平均值:

$$(a) f(x) = \sin^2 x + \cos^2(x\sqrt{2});$$

$$(b) f(x) = \operatorname{arc} \operatorname{tg} x; \quad (B) f(x) = \sqrt{x} \sin x.$$

解 (a) 由于

$$\begin{aligned} & \int_0^x [\sin^2 \xi + \cos^2(\xi\sqrt{2})] d\xi \\ &= \int_0^x \left[\frac{1 - \cos 2\xi}{2} + \frac{1 + \cos(2\xi\sqrt{2})}{2} \right] d\xi \\ &= x - \frac{1}{4} \sin 2x + \frac{1}{4\sqrt{2}} \sin(2x\sqrt{2}), \end{aligned}$$

所以

$$\begin{aligned} M[f] &= \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x [\sin^2 \xi + \cos^2(\xi\sqrt{2})] d\xi \\ &= \lim_{x \rightarrow +\infty} \left[1 - \frac{1}{4x} \sin 2x \right. \\ & \quad \left. + \frac{1}{4x\sqrt{2}} \sin(2x\sqrt{2}) \right] \\ &= 1; \end{aligned}$$

$$(b) M[f] = \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x \operatorname{arc} \operatorname{tg} \xi d\xi$$

$$= \lim_{x \rightarrow +\infty} \frac{1}{x} \left[x \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \ln(1+x^2) \right]$$

$$\begin{aligned}
&= \frac{\pi}{2} - \lim_{x \rightarrow +\infty} \frac{\ln(1+x^2)}{2x} \\
&= \frac{\pi}{2} - \lim_{x \rightarrow +\infty} \frac{2x}{2(1+x^2)} = \frac{\pi}{2};
\end{aligned}$$

(B) 利用第二中值定理, 得

$$\int_0^x \sqrt{\xi} \sin \xi \, d\xi = \sqrt{c} \int_0^x \sin \xi \, d\xi$$

$$= \sqrt{c} (\cos c - \cos x) \quad (0 \leq c \leq x),$$

于是,

$$\begin{aligned}
M[f] &= \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x \sqrt{\xi} \sin \xi \, d\xi \\
&= \lim_{x \rightarrow +\infty} \frac{\cos c - \cos x}{\sqrt{x}} = 0.
\end{aligned}$$

2357. 求:

$$(a) \lim_{x \rightarrow 0} x \int_x^1 \frac{\cos t}{t^2} \, dt; \quad (b) \lim_{x \rightarrow \infty} \frac{\int_0^x \sqrt{1+t^4} \, dt}{x^3};$$

$$(c) \lim_{x \rightarrow +0} \frac{\int_x^{+\infty} t^{-1} e^{-t} \, dt}{\ln \frac{1}{x}};$$

$$(d) \lim_{x \rightarrow +0} x^a \int_x^1 \frac{f(t)}{t^{a+1}} \, dt,$$

其中 $a > 0$, $f(t)$ 为闭区间 $[0, 1]$ 上的连续函数.

解 (a) 由于

$$1 - \frac{t^2}{2} \leq \cos t \leq 1,$$

所以

$$\int_x^1 \frac{1 - \frac{t^2}{2}}{t^2} dt \leq \int_x^1 \frac{\cos t}{t^2} dt \leq \int_x^1 \frac{dt}{t^2},$$

计算得

$$-\frac{3}{2} + \frac{x}{2} + \frac{1}{x} \leq \int_x^1 \frac{\cos t}{t^2} dt \leq -1 + \frac{1}{x}.$$

又由于

$$\lim_{x \rightarrow 0} x \left(-\frac{3}{2} + \frac{x}{2} + \frac{1}{x} \right) = 1$$

及

$$\lim_{x \rightarrow 0} x \left(-1 + \frac{1}{x} \right) = 1,$$

故最后得到

$$\lim_{x \rightarrow 0} x \int_x^1 \frac{\cos t}{t^2} dt = 1;$$

(6) 由于

$$t^2 < \sqrt{1+t^4},$$

所以

$$\int_0^x \sqrt{1+t^4} dt > \int_0^x t^2 dt = \frac{x^3}{3},$$

从而当 $x \rightarrow +\infty$ 时, $\int_0^x \sqrt{1+t^4} dt \rightarrow +\infty$.

利用洛比塔法则, 得

$$\lim_{x \rightarrow +\infty} \frac{\int_0^x \sqrt{1+t^4} dt}{x^3} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1+x^4}}{3x^2} = \frac{1}{3};$$

(B) 由于 $\lim_{t \rightarrow +0} f \cdot (t^{-1} e^{-t}) = 1$ ，故广义积分

$\int_0^{+\infty} t^{-1} e^{-t} dt$ 发散，从而，所求的极限是 $\frac{\infty}{\infty}$ 型未定

式。利用洛比塔法则，得

$$\lim_{x \rightarrow +0} \frac{\int_x^{+\infty} t^{-1} e^{-t} dt}{\ln \frac{1}{x}} = \lim_{x \rightarrow +0} \frac{-e^{-x} \cdot x^{-1}}{-\frac{1}{x}} = 1;$$

(r) 由于 $f(t)$ 在 $t=0$ 处右连续，故对于任意给定的 $\varepsilon > 0$ ，总存在 $\delta' > 0$ ，使当 $0 < t < \delta'$ 时，恒有

$$|f(t) - f(0)| < \frac{\alpha \varepsilon}{2}.$$

今又取 $0 < \delta < \delta'$ ，使当 $0 < x < \delta$ 时，有

$$\left| x^\alpha \int_0^1 \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right| < \frac{\varepsilon}{2}.$$

于是，当 $0 < x < \delta$ 时，就有

$$\begin{aligned} & \left| x^\alpha \int_x^1 \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right| \\ &= \left| x^\alpha \int_x^{\delta'} \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right. \\ & \quad \left. + x^\alpha \int_{\delta'}^1 \frac{f(t) - f(0)}{t^{\alpha+1}} dt \right| \\ &\leq \frac{\alpha \varepsilon}{2} \cdot x^\alpha \int_x^{\delta'} \frac{dt}{t^{\alpha+1}} + \frac{\varepsilon}{2} \end{aligned}$$

$$= \frac{\varepsilon}{2} x^\alpha \left(\frac{1}{x^\alpha} - \frac{1}{\delta^{\alpha+1}} \right) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

故

$$\lim_{x \rightarrow +0} x^\alpha \int_x^1 \frac{f(t) - f(0)}{t^{\alpha+1}} dt = 0,$$

最后得到

$$\begin{aligned} \lim_{x \rightarrow +0} x^\alpha \int_x^1 \frac{f(t)}{t^{\alpha+1}} dt &= \lim_{x \rightarrow +0} x^\alpha \int_x^1 \frac{f(0)}{t^{\alpha+1}} dt \\ &= \lim_{x \rightarrow +0} x^\alpha f(0) \left[-\frac{1}{\alpha} t^{-\alpha} \right] \Big|_x^1 \\ &= \lim_{x \rightarrow +0} x^\alpha f(0) \left(\frac{1}{\alpha x^\alpha} - \frac{1}{\alpha} \right) = \frac{f(0)}{\alpha}. \end{aligned}$$

*) 原题(B) (Γ)中 $x \rightarrow +0$ 误印为 $x \rightarrow 0$.
研究下列积分的收敛性:

$$2358. \int_0^{+\infty} \frac{x^2 dx}{x^4 - x^2 + 1}.$$

解 由于 $x^2 \frac{x^2}{x^4 - x^2 + 1} \rightarrow 1$ (当 $x \rightarrow +\infty$ 时),

所以积分 $\int_0^{+\infty} \frac{x^2 dx}{x^4 - x^2 + 1}$ 收敛.

$$2359. \int_1^{+\infty} \frac{dx}{x^{\frac{5}{3}} \sqrt{x^2 + 1}}.$$

解 由于 $x^{\frac{5}{3}} \cdot \frac{1}{x^{\frac{5}{3}} \sqrt{x^2 + 1}} \rightarrow 1$ (当 $x \rightarrow +\infty$ 时),

所以积分 $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2+1}}$ 收敛.

$$2360. \int_0^2 \frac{dx}{\ln x}.$$

解 当 $0 < x < 1$ 时 $\ln x < 0$. 由于

$$\lim_{x \rightarrow 1-0} (1-x) \cdot \frac{1}{-\ln x} = \lim_{x \rightarrow 1-0} \frac{-1}{-\frac{1}{x}} = 1,$$

所以积分 $\int_0^1 \frac{dx}{\ln x}$ 发散, 从而积分 $\int_0^2 \frac{dx}{\ln x}$ 也发散.

$$2361. \int_0^{+\infty} x^{p-1} e^{-x} dx.$$

解 将积分分成 $\int_0^{+\infty} x^{p-1} e^{-x} dx = \int_0^1 x^{p-1} e^{-x} dx$
 $+ \int_1^{+\infty} x^{p-1} e^{-x} dx.$

对于积分 $\int_0^1 x^{p-1} e^{-x} dx$. 由于

$$x^{1-p} \cdot (x^{p-1} e^{-x}) \rightarrow 1 \quad (\text{当 } x \rightarrow +0 \text{ 时}),$$

故当 $p > 0$ 时 (从而 $1-p < 1$), 积分 $\int_0^1 x^{p-1} e^{-x} dx$ 收敛.

对于积分 $\int_1^{+\infty} x^{p-1} e^{-x} dx$. 由于

$$x^2 \cdot (x^{p-1} e^{-x}) = \frac{x^{p+1}}{e^x} \rightarrow 0 \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故对于一切 p 值, 积分 $\int_1^{+\infty} x^{p-1} e^{-x} dx$ 恒收敛.

于是, 当 $p > 0$ 时, 积分

$$\int_0^{+\infty} x^{p-1} e^{-x} dx$$

收敛.

2362. $\int_0^1 x^p \ln^q \frac{1}{x} dx.$

解 将积分分成 $\int_0^1 x^p \ln^q \frac{1}{x} dx = \int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$
 $+ \int_{\frac{1}{2}}^1 x^p \ln^q \frac{1}{x} dx.$

对于积分 $\int_{\frac{1}{2}}^1 x^p \ln^q \frac{1}{x} dx$, 由于

$$\lim_{x \rightarrow 1-0} (1-x)^{-q} \cdot x^p \ln^q \frac{1}{x} = \left(\lim_{x \rightarrow 1-0} x^p \left(\frac{\ln \frac{1}{x}}{1-x} \right)^q \right)$$
$$= \left(\lim_{x \rightarrow 1-0} \frac{\ln \frac{1}{x}}{1-x} \right)^q = \left(\lim_{x \rightarrow 1-0} \frac{x \left(-\frac{1}{x^2} \right)}{-1} \right)^q = 1,$$

故 $\int_{\frac{1}{2}}^1 x^p \ln^q \frac{1}{x} dx$ 当 $-q < 1$ (即 $q > -1$) 时收敛,

当 $-q \geq 1$ (即 $q \leq -1$) 时发散. 于是, 当 $q \leq -1$

时, $\int_0^1 x^p \ln^q \frac{1}{x} dx$ 必发散. 故下面可在 $q > -1$ 的假

定下来讨论 $\int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$.

若 $p > -1$, 可取 $\tau > 0$ 充分小, 使 $p - \tau > -1$.
于是

$$\lim_{x \rightarrow +0} x^{-p+\tau} \cdot x^p \ln^q \frac{1}{x} = \lim_{x \rightarrow +0} \frac{\left(\ln \frac{1}{x}\right)^q}{\left(\frac{1}{x}\right)^\tau} = 0.$$

由于 $p + \tau < 1$, 故此时 $\int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$ 收敛;
若 $p \leq -1$ (设 $q > -1$), 则

$$\begin{aligned} \int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx &\geq \int_0^{\frac{1}{2}} x^{-1} \ln^q \frac{1}{x} dx \\ &= - \int_0^{\frac{1}{2}} \left(\ln \frac{1}{x}\right)^q d\left(\ln \frac{1}{x}\right) \\ &= - \frac{\left(\ln \frac{1}{x}\right)^{q+1}}{q+1} \Big|_0^{\frac{1}{2}} = +\infty, \end{aligned}$$

故此时 $\int_0^{\frac{1}{2}} x^p \ln^q \frac{1}{x} dx$ 发散.

总之, 仅当 $p > -1$ 且 $q > -1$ 时积分

$$\int_0^1 x^p \ln^q \frac{1}{x} dx \text{ 收敛.}$$

2363. $\int_0^{+\infty} \frac{x^n}{1+x^n} dx \quad (n \geq 0).$

解 先考虑积分 $\int_0^1 \frac{x^n}{1+x^n} dx$. 由于

$$x^{-n} \cdot \frac{x^n}{1+x^n} \rightarrow 1 \quad (\text{当 } x \rightarrow +0 \text{ 时}),$$

故积分 $\int_0^1 \frac{x^m}{1+x^n} dx$ 仅当 $-m < 1$, 即仅当 $m > -1$ 时收敛.

再考虑积分 $\int_1^{+\infty} \frac{x^m}{1+x^n} dx$. 由于

$$x^{n-m} \cdot \frac{x^m}{1+x^n} \longrightarrow 1 \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故积分 $\int_1^{+\infty} \frac{x^m}{1+x^n} dx$ 仅当 $n-m > 1$ 时收敛.

于是, 当 $m > -1$ 且 $n-m > 1$ 时, 积分

$$\int_0^{+\infty} \frac{x^m}{1+x^n} dx \quad (n \geq 0)$$

收敛.

2364. $\int_0^{+\infty} \frac{\operatorname{arc} \operatorname{tg} ax}{x^n} dx \quad (a \neq 0).$

解 由于 $\operatorname{arc} \operatorname{tg} ax = -\operatorname{arc} \operatorname{tg} (-ax)$, 故可设 $a > 0$,

先考虑积分 $\int_0^1 \frac{\operatorname{arc} \operatorname{tg} ax}{x^n} dx$. 由于

$$\begin{aligned} \lim_{x \rightarrow +0} x^{n-1} \cdot \frac{\operatorname{arc} \operatorname{tg} ax}{x^n} &= \lim_{x \rightarrow +0} \frac{\operatorname{arc} \operatorname{tg} ax}{x} \\ &= \lim_{x \rightarrow +0} \frac{\frac{a}{1+a^2x^2}}{1} = a, \end{aligned}$$

故积分 $\int_0^1 \frac{\operatorname{arc} \operatorname{tg} ax}{x^n} dx$ 仅当 $n-1 < 1$ 即当 $n < 2$ 时收敛.

再考虑积分 $\int_1^{+\infty} \frac{\operatorname{arc\,tg} ax}{x^n} dx$. 由于

$$x^n \cdot \frac{\operatorname{arc\,tg} ax}{x^n} \rightarrow \frac{\pi}{2} \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故积分 $\int_1^{+\infty} \frac{\operatorname{arc\,tg} ax}{x^n} dx$ 仅当 $n > 1$ 时收敛.

于是, 仅当 $1 < n < 2$ 时, 积分

$$\int_0^{+\infty} \frac{\operatorname{arc\,tg} ax}{x^n} dx \quad (a \neq 0)$$

收敛.

2365. $\int_0^{+\infty} \frac{\ln(1+x)}{x^n} dx$.

解 先考虑积分 $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$. 当 $n > 1$ 时, 取

$a > 0$ 充分小, 使 $n - a > 1$. 由于

$$x^{n-a} \cdot \frac{\ln(1+x)}{x^n} = \frac{\ln(1+x)}{x^a} \rightarrow 0$$

(当 $x \rightarrow +\infty$ 时),

故此时积分 $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$ 收敛. 当 $n \leq 1$ 时,

由于

$$x^n \cdot \frac{\ln(1+x)}{x^n} \rightarrow +\infty \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故此时积分 $\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$ 发散.

再考虑积分 $\int_0^1 \frac{\ln(1+x)}{x^n} dx$ 。由于

$$\lim_{x \rightarrow +0} x^{n-1} \cdot \frac{\ln(1+x)}{x^n} = \lim_{x \rightarrow +0} \frac{\ln(1+x)}{x} = 1,$$

故积分 $\int_0^1 \frac{\ln(1+x)}{x^n} dx$ 仅当 $n-1 < 1$ 即当 $n < 2$ 时

收敛。

于是，仅当 $1 < n < 2$ 时，积分

$$\int_0^{+\infty} \frac{\ln(1+x)}{x^n} dx$$

收敛。

2366. $\int_0^{+\infty} \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx \quad (n \geq 0).$

解 先考虑积分 $\int_0^1 \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$ 。由于

$$\lim_{x \rightarrow +0} x^{-m-1} \cdot \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} = \frac{1}{2} \lim_{x \rightarrow +0} \frac{\operatorname{arc} \operatorname{tg} x}{x}$$

$$= \frac{1}{2} \lim_{x \rightarrow +0} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{2},$$

故积分 $\int_0^1 \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$ 仅当 $-m-1 < 1$ 即当

$m > -2$ 时收敛。

再考虑积分 $\int_1^{+\infty} \frac{x^m \operatorname{arc} \operatorname{tg} x}{2+x^n} dx$ 。由于

$$x^{n-m} \cdot \frac{x^m \operatorname{arctg} x}{2+x^2}$$

$$= \frac{x^n \operatorname{arctg} x}{2+x^2} \rightarrow \frac{\pi}{2} \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故积分 $\int_1^{+\infty} \frac{x^m \operatorname{arctg} x}{2+x^2} dx$ 仅当 $n-m > 1$ 时收敛。

于是, 仅当 $m > -2$ 且 $n-m > 1$ 时, 积分

$$\int_0^{+\infty} \frac{x^m \operatorname{arctg} x}{2+x^2} dx \quad (n \geq 0)$$

收敛。

2367. $\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx \quad (n \geq 0).$

解 当 $a \neq 0$ 时, 设 $f(x) = \cos ax$, $g(x) = \frac{1}{1+x^n}$,

则对于任意的 $A > 0$, 均有 $\left| \int_0^A f(x) dx \right| \leq \frac{2}{a}$; 其次,

当 $n > 0$ 时, $g(x)$ 单调下降且趋于零 ($n \rightarrow +\infty$).

从而得知积分

$$\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx$$

收敛。至于当 $n = 0$ 时, 积分显然发散。

当 $a = 0$ 时, 由于

$$x^n \cdot \frac{1}{1+x^n} \rightarrow 1 \quad (\text{当 } x \rightarrow +\infty \text{ 时}),$$

故积分 $\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx$ 仅当 $n > 1$ 时收敛。

于是, 当 $a \neq 0$ 、 $n > 0$ 及 $a = 0$ 、 $n > 1$ 时, 积分

$$\int_0^{+\infty} \frac{\cos ax}{1+x^n} dx.$$

收敛.

2368. $\int_0^{+\infty} \frac{\sin^2 x}{x} dx.$

解 方法一:

$$\frac{\sin^2 x}{x} = \frac{1 - \cos 2x}{2x} = \frac{1}{2} \left(\frac{1}{x} - \frac{\cos 2x}{x} \right).$$

积分 $\int_1^{+\infty} \frac{dx}{x}$ 显然发散.

又因对于任意的 $A > 1$, $\left| \int_1^A \cos 2x dx \right| \leq 2$,

且当 $x \rightarrow +\infty$ 时, $\frac{1}{x}$ 单调地趋于零, 故积分

$$\int_1^{+\infty} \frac{\cos 2x}{x} dx \text{ 收敛.}$$

于是, 积分 $\int_1^{+\infty} \frac{\sin^2 x}{x} dx$ 发散, 从而积分

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx$$

发散.

方法二:

$$\begin{aligned} \int_0^{+\infty} \frac{\sin^2 x}{x} dx &= \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin^2 x}{x} dx \\ &= \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin^2 t}{t+n\pi} dt \geq \frac{1}{\pi} \int_0^{\pi} \sin^2 t dt \cdot \sum_{n=0}^{\infty} \frac{1}{n+1} \end{aligned}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n},$$

由于不论 N 取多大, 只要取 $p=N$, 就有

$$\begin{aligned} \sum_{k=N+1}^{N+p} \frac{1}{k} &= \sum_{k=N+1}^{2N} \frac{1}{k} \\ &= \frac{1}{N+1} + \cdots + \frac{1}{2N} > \underbrace{\frac{1}{2N} + \frac{1}{2N} + \cdots + \frac{1}{2N}}_{N \text{ 个}} \\ &= \frac{1}{2N} \cdot N = \frac{1}{2}, \end{aligned}$$

故递增叙列

$$S_n = \sum_{k=1}^n \frac{1}{k} \quad (n=1, 2, \dots)$$

的极限 $\lim_{n \rightarrow \infty} S_n$ 是 $+\infty$, 即 $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$.

于是, 积分

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx$$

发散.

2369. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$.

解 先考虑积分 $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$. 对于任何 q 值,

由于

$$\lim_{x \rightarrow 0} x^q \cdot \frac{1}{\sin^p x \cos^q x}$$

$$= \lim_{x \rightarrow +0} \left(\frac{x}{\sin x} \right)^p \cdot \lim_{x \rightarrow +0} \left(\frac{1}{\cos^q x} \right) = 1,$$

故积分 $\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^p x \cos^q x}$ 仅当 $p < 1$ (q 为任意值) 时收敛。

再考虑积分 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$ 。对于任何 p 值，

由于

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}-0} \left(\frac{\pi}{2} - x \right)^q \cdot \frac{1}{\sin^p x \cos^q x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}-0} \left(\frac{\frac{\pi}{2} - x}{\cos x} \right)^q \cdot \lim_{x \rightarrow \frac{\pi}{2}-0} \left(\frac{1}{\sin^p x} \right) \\ &= \lim_{t \rightarrow +0} \left(\frac{t}{\sin t} \right)^q = 1, \end{aligned}$$

故积分 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$ 仅当 $q < 1$ (p 为任意值) 时收敛。

于是，当 $p < 1$ 且 $q < 1$ 时，积分

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin^p x \cos^q x}$$

收敛。

2370. $\int_0^1 \frac{x^a dx}{\sqrt{1-x^2}}$.

解 先考虑积分 $\int_0^{\frac{1}{2}} \frac{x^n dx}{\sqrt{1-x^2}}$. 由于

$$\lim_{x \rightarrow +0} \left(x^{-n} \cdot \frac{x^n}{\sqrt{1-x^2}} \right) = 1,$$

故积分 $\int_0^{\frac{1}{2}} \frac{x^n dx}{\sqrt{1-x^2}}$ 仅当 $-n < 1$ 即当 $n > -1$ 时收敛.

再考虑积分 $\int_{\frac{1}{2}}^1 \frac{x^n}{\sqrt{1-x^2}} dx$. 对于任意的 n , 由于

$$\begin{aligned} & \lim_{x \rightarrow 1-0} \left(\sqrt{1-x} \cdot \frac{x^n}{\sqrt{1-x^2}} \right) \\ &= \lim_{x \rightarrow 1-0} \frac{x^n}{\sqrt{1+x}} = \frac{1}{\sqrt{2}}, \end{aligned}$$

故积分 $\int_{\frac{1}{2}}^1 \frac{x^n}{\sqrt{1-x^2}} dx$ 恒收敛.

于是, 当 $n > -1$ 时, 积分

$$\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}$$

收敛.

2371. $\int_0^{+\infty} \frac{dx}{x^p + x^q}$.

解 先考虑积分 $\int_0^1 \frac{dx}{x^p + x^q}$. 不妨设 $\min(p, q) = p$,

由于

$$\lim_{x \rightarrow +0} \left(x^p \cdot \frac{1}{x^p + x^q} \right) = \lim_{x \rightarrow +0} \frac{1}{1 + x^{q-p}} = 1,$$

故积分 $\int_0^1 \frac{dx}{x^p + x^q}$ 仅当 $p < 1$ 即当 $\min(p, q) < 1$ 时收敛。

再考虑积分 $\int_1^{+\infty} \frac{dx}{x^p + x^q}$ 。不妨设 $\max(p, q) = q$ ，由于

$$\lim_{x \rightarrow +\infty} \left(x^q \cdot \frac{1}{x^p + x^q} \right) = \lim_{x \rightarrow +\infty} \frac{1}{x^{-(q-p)} + 1} = 1,$$

故积分 $\int_1^{+\infty} \frac{dx}{x^p + x^q}$ 仅当 $q > 1$ 即当 $\max(p, q) > 1$ 时收敛。

于是，当 $\min(p, q) < 1$ 且 $\max(p, q) > 1$ 时，积分

$$\int_0^{+\infty} \frac{dx}{x^p + x^q}$$

收敛。

2372. $\int_0^1 \frac{\ln x}{1-x^2} dx.$

解 先考虑积分 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x^2} dx$ 。由于

$$\lim_{x \rightarrow +0} \left(\sqrt{x} \cdot \frac{\ln x}{1-x^2} \right) = 0,$$

故积分 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x^2} dx$ 收敛。

再考虑积分 $\int_{\frac{1}{2}}^1 \frac{\ln x}{1-x^2} dx$. 由于

$$\lim_{x \rightarrow 1-0} \left(\sqrt{1-x} \cdot \frac{\ln x}{1-x^2} \right) = 0,$$

故积分 $\int_{\frac{1}{2}}^1 \frac{\ln x}{1-x^2} dx$ 收敛.

于是, 积分

$$\int_0^1 \frac{\ln x}{1-x^2} dx$$

收敛.

2373. $\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\sqrt{x}} dx.$

解 由于

$$\begin{aligned} & \lim_{x \rightarrow +0} \left[x^{\frac{5}{6}} \cdot \frac{\ln(\sin x)}{\sqrt{x}} \right] \\ &= \lim_{x \rightarrow +0} \left[\left(\frac{x}{\sin x} \right)^{\frac{1}{3}} \cdot \sqrt[3]{\sin x} \ln(\sin x) \right] = 0, \end{aligned}$$

故积分 $\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\sqrt{x}} dx$ 收敛.

2374. $\int_1^{+\infty} \frac{dx}{x^p \ln^q x}.$

解 先考虑 $\int_1^2 \frac{dx}{x^p \ln^q x}$. 对于任意的 p , 由于

$$\lim_{x \rightarrow 1+0} \left[(x-1)^q \cdot \frac{1}{x^p \ln^q x} \right]$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1+0} \left[\frac{1}{x^p} \cdot \left(\frac{x-1}{\ln x} \right)^q \right] = \left(\lim_{x \rightarrow 1+0} \frac{x-1}{\ln x} \right)^q \\
&= \left(\lim_{x \rightarrow 1+0} \frac{1}{\frac{1}{x}} \right)^q = 1,
\end{aligned}$$

故积分 $\int_1^2 \frac{dx}{x^p \ln^q x}$ 仅当 $q < 1$ 且 p 为任意值时收敛。

再考虑积分 $\int_2^{+\infty} \frac{dx}{x^p \ln^q x}$ 。如果 $p > 1$ ，取 $\alpha > 0$

充分小，使 $p - \alpha > 1$ ，则对于任意的 q ，由于

$$\lim_{x \rightarrow +\infty} \left(x^{p-\alpha} \cdot \frac{1}{x^p \ln^q x} \right) = \lim_{x \rightarrow +\infty} \left(\frac{1}{x^\alpha \ln^q x} \right) = 0,$$

故积分 $\int_2^{+\infty} \frac{dx}{x^p \ln^q x}$ 收敛；如果 $p \leq 1$ ， $q < 1$ ，由于

$$\begin{aligned}
&\int_2^{+\infty} \frac{dx}{x^p \ln^q x} \geq \int_2^{+\infty} \frac{dx}{x \ln^q x} \\
&= \frac{(\ln x)^{1-q}}{1-q} \Big|_2^{+\infty} = +\infty,
\end{aligned}$$

故积分 $\int_2^{+\infty} \frac{dx}{x^p \ln^q x}$ 发散。

于是，当 $p > 1$ 且 $q < 1$ 时，积分

$$\int_1^{+\infty} \frac{dx}{x^p \ln^q x}$$

收敛。

$$2375. \int_e^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}.$$

解 先考虑积分 $\int_e^3 \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$. 对于任意的 p 和 q , 由于

$$\begin{aligned} & \lim_{x \rightarrow e+0} \frac{(x-e)^r}{x^p (\ln x)^q (\ln \ln x)^r} \\ &= \frac{1}{e^p} \left(\lim_{x \rightarrow e+0} \frac{x-e}{\ln \ln x} \right)^r \\ &= \frac{1}{e^p} \left(\lim_{x \rightarrow e+0} \frac{1}{\frac{1}{x \ln x}} \right)^r = e^{r-p}, \end{aligned}$$

故积分 $\int_e^3 \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$ 仅当 $r < 1$ 和 p, q 为任意值时收敛.

再考虑积分 $\int_3^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$. 分三种情形讨论:

(1) 如果 $p > 1$, q 和 r 为任意值. 取 $\alpha > 0$ 充分小, 使 $p - \alpha > 1$. 由于

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{x^{p-\alpha}}{x^p (\ln x)^q (\ln \ln x)^r} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{x^\alpha (\ln x)^q (\ln \ln x)^r} = 0, \end{aligned}$$

故此时积分 $\int_3^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$ 收敛,

(2) 当 $p = 1$ 时, 则有

$$\int_s^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$$

$$= \int_{1.3}^{+\infty} \frac{dx}{x^q (\ln x)^r},$$

利用2374题的结果得知, 当 $p=1$, $q>1$ 和 $r<1$ 时

积分 $\int_s^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$ 收敛;

(3) 当 $p<1$ 时, 取 $\delta>0$ 充分小, 使 $p+\delta<1$. 对于任意的 q 和 r , 由于

$$\lim_{x \rightarrow +\infty} \frac{x^{p+\delta}}{x^p (\ln x)^q (\ln \ln x)^r}$$

$$= \lim_{x \rightarrow +\infty} \frac{x^\delta}{(\ln x)^q (\ln \ln x)^r} = +\infty,$$

故此时积分 $\int_s^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$ 发散.

于是, 当 $p>1$, q 是任意的, $r<1$ 和当 $p=1$, $q>1$, $r<1$ 时, 积分

$$\int_s^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln \ln x)^r}$$

收敛.

2376.
$$\int_{-\infty}^{+\infty} \frac{dx}{|x-a_1|^{p_1} |x-a_2|^{p_2} \cdots |x-a_n|^{p_n}}.$$

解 首先, 被积函数关于 $\frac{1}{x}$ 是 $\sum_{i=1}^n p_i$ 级无穷小 (当 $x \rightarrow \pm\infty$ 时),

其次 (不妨设当 $i \neq j$ 时, $a_i \neq a_j$),

$$\lim_{x \rightarrow a_i} \left[|x - a_i|^{-p_i} \cdot \frac{1}{|x - a_1|^{p_1} |x - a_2|^{p_2} \cdots |x - a_n|^{p_n}} \right]$$

$$= c_i, \quad 0 < c_i < +\infty \quad (i = 1, 2, \dots, n),$$

故积分 $\int_{-\infty}^{+\infty} \frac{dx}{|x - a_1|^{p_1} |x - a_2|^{p_2} \cdots |x - a_n|^{p_n}}$ 仅当

$$\sum_{i=1}^n p_i > 1 \text{ 且 } p_i < 1 \quad (i = 1, 2, \dots, n) \text{ 时收敛.}$$

2377. $\int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx,$

式中 $P_m(x)$ 及 $P_n(x)$ 为次数分别为 m 及 n 的互质的多项式.

解 当 $P_n(x) = 0$ 在 $(0, +\infty)$ 上有根 λ 并设其重数为 $r (\geq 1)$ 时, 由于 $P_n(x)$ 与 $P_m(x)$ 互质, 故 λ 不是 $P_m(x)$ 的根. 从而有

$$\lim_{x \rightarrow \lambda} \left[(x - \lambda)^r \cdot \frac{P_m(x)}{P_n(x)} \right] = a \neq 0,$$

而且显然在点 λ 的右 (左) 近旁, $\frac{P_m(x)}{P_n(x)}$ 都保持定号. 由于 $r \geq 1$, 故积分发散. 由于

$$\lim_{x \rightarrow +\infty} \left[x^{n-m} \cdot \frac{P_m(x)}{P_n(x)} \right] = b \neq 0,$$

故积分 $\int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx$ 仅当 $n - m > 1$ 即当 $n > m + 1$

时收敛。

于是，当 $P_n(x)$ 在区间 $[0, +\infty)$ 内无根且 $n > m + 1$ 时，积分

$$\int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx$$

收敛。

研究下列积分的绝对收敛性和条件收敛性：

2378. $\int_0^{+\infty} \frac{\sin x}{x} dx.$

解 对于任意的 $A > 1$ ，由于 $\left| \int_1^A \sin x dx \right| \leq 2$ ，且

当 $x \rightarrow +\infty$ 时， $\frac{1}{x}$ 单调地趋于零，故积分

$$\int_1^{+\infty} \frac{\sin x}{x} dx$$

收敛。而积分 $\int_0^1 \frac{\sin x}{x} dx$ 是普通的定积分（ $\frac{\sin x}{x}$ 在 $x = 0$ 有可去间断点，故补充定义其值为 1 后， $\frac{\sin x}{x}$ 可视为 $[0, 1]$ 上的连续函数），故积分

$$\int_0^{+\infty} \frac{\sin x}{x} dx$$

收敛 但它不是绝对收敛的。事实上，当 $x > 0$ 时，

$\left| \frac{\sin x}{x} \right| \geq \frac{\sin^2 x}{x}$ ，由 2368 题知，积分 $\int_0^{+\infty} \frac{\sin^2 x}{x} dx$

发散，故积分 $\int_0^{+\infty} \left| \frac{\sin x}{x} \right| dx$ 发散。

$$2379. \int_0^{+\infty} \frac{\sqrt{x} \cos x}{x+100} dx.$$

解 对于任意的 $A > 0$, 由于 $\left| \int_0^A \cos x dx \right| \leq 2$, 且

当 $x \rightarrow +\infty$ 时, $\frac{\sqrt{x}}{x+100}$ 单调地趋于零, 故积分

$$\int_0^{+\infty} \frac{\sqrt{x} \cos x}{x+100} dx$$

收敛. 但它不是绝对收敛的. 事实上, 由于

$$\begin{aligned} \frac{\sqrt{x} |\cos x|}{x+100} &\geq \frac{\sqrt{x} \cos^2 x}{x+100} \\ &= \frac{1}{2} \left(\frac{\sqrt{x}}{x+100} - \frac{\sqrt{x} \cos 2x}{x+100} \right), \end{aligned}$$

且 $\lim_{x \rightarrow +\infty} \left(x^{\frac{1}{2}} \cdot \frac{\sqrt{x}}{x+100} \right) = 1$, 故积分 $\int_0^{+\infty} \frac{\sqrt{x}}{x+100} dx$

发散. 仿照前半段证明, 可知 $\int_0^{+\infty} \frac{\sqrt{x} \cos 2x}{x+100} dx$

收敛. 从而, 积分 $\int_0^{+\infty} \frac{\sqrt{x} \cos^2 x}{x+100} dx$ 发散. 于是,

积分

$$\int_0^{+\infty} \frac{\sqrt{x} |\cos x|}{x+100} dx$$

发散.

$$2380. \int_0^{+\infty} x^q \sin(x^q) dx \quad (q \neq 0).$$

解 设 $t=x^q$, 则 $dx=\frac{1}{q}t^{\frac{1}{q}-1}dt$. 于是

$$\int_0^{+\infty} x^p \sin(x^q) dx = \frac{1}{|q|} \int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt.$$

先考虑积分 $\int_0^1 t^{\frac{p+1}{q}-1} \sin t dt$. 由于

$$\lim_{t \rightarrow +0} \left(t^{-\frac{p+1}{q}} \cdot t^{\frac{p+1}{q}-1} \sin t \right) = \lim_{t \rightarrow +0} \frac{\sin t}{t} = 1,$$

故积分 $\int_0^1 t^{\frac{p+1}{q}-1} \sin t dt$ 仅当 $-\frac{p+1}{q} < 1$, 即当

$\frac{p+1}{q} > -1$ 时收敛, 又由于被积函数在 $[0, 1]$ 上

非负, 故也是绝对收敛的.

再考虑积分 $\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$. 如果 $-\frac{p+1}{q} < 1$,

则由于对任意的 $A > 1$, $\left| \int_1^A \sin t dt \right| \leq 2$ 且 $t^{\frac{p+1}{q}-1}$

单调地趋于零 (当 $t \rightarrow +\infty$ 时), 故此时积分

$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$ 收敛. 如果 $-\frac{p+1}{q} = 1$, 则积分

$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt$ 显然发散, 从而积分 $\int_0^{+\infty} t^{\frac{p+1}{q}-1}$

$\sin t dt$ 也发散. 如果 $-\frac{p+1}{q} > 1$, 则由于 $\lim_{t \rightarrow +\infty} t^{\frac{p+1}{q}-1}$

$= +\infty$, 故对任给的 $A > 0$, 总存在自然数 N , 使有

$2N\pi + \frac{\pi}{4} > A$, 且当 $t > 2N\pi + \frac{\pi}{4}$ 时, $t^{\frac{p+1}{q}-1} > \sqrt{2}$.

今取

$$A' = 2N\pi + \frac{\pi}{4}, \quad A'' = 2N\pi + \frac{\pi}{2},$$

则有

$$\left| \int_{A'}^{A''} t^{\frac{p+1}{q}-1} \sin t \, dt \right| > \sqrt{2} \left| \int_{A'}^{A''} \sin t \, dt \right| \\ = 1,$$

它不可能小于任给的 ε ($0 < \varepsilon < 1$), 因而, 积分

$$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t \, dt,$$

发散, 从而积分

$$\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t \, dt$$

也发散.

于是, 仅当 $-1 < -\frac{p+1}{q} < 1$ 时, 积分

$$\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t \, dt$$

收敛, 且当 $\frac{p+1}{q} > -1$ 时, 积分

$$\int_0^1 t^{\frac{p+1}{q}-1} \sin t \, dt$$

绝对收敛.

下面我们考虑积分 $\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t \, dt$ 的绝对收

敛性，分三种情形讨论：

(1) 当 $\frac{p+1}{q} < 0$ 时，由于

$$|t^{\frac{p+1}{q}-1} \sin t| \leq t^{\frac{p+1}{q}-1} \quad (1 \leq t < +\infty),$$

且 $\int_1^{+\infty} t^{\frac{p+1}{q}-1} dt$ 收敛，故当 $\frac{p+1}{q} < 0$ 时，积分

$$\int_1^{+\infty} t^{\frac{p+1}{q}-1} \sin t dt \text{ 绝对收敛；}$$

(2) 当 $\frac{p+1}{q} = 0$ 时，由于

$$\begin{aligned} & \int_1^{+\infty} |t^{\frac{p+1}{q}-1} \sin t| dt \\ &= \int_1^{+\infty} \left| \frac{\sin t}{t} \right| dt = +\infty, \end{aligned}$$

故此时积分不绝对收敛(但条件收敛)；

(3) 当 $\frac{p+1}{q} > 0$ 时，由于

$$\begin{aligned} & \int_1^{+\infty} |t^{\frac{p+1}{q}-1} \sin t| dt \\ & \geq \int_1^{+\infty} \frac{|\sin t|}{t} dt = +\infty, \end{aligned}$$

故此时积分也不是绝对收敛的。

于是，当 $-1 < \frac{p+1}{q} < 0$ 时，积分

$$\int_0^{+\infty} t^{\frac{p+1}{q}-1} \sin t \, dt$$

绝对收敛。

最后我们得到：当 $-1 < \frac{p+1}{q} < 0$ 时，积分

$$\int_0^{+\infty} x^p \sin(x^q) dx$$

绝对收敛；当 $0 \leq \frac{p+1}{q} < 1$ 时，积分条件收敛。

2381. $\int_0^{+\infty} \frac{x^p \sin x}{1+x^q} dx \quad (q \geq 0).$

解 先考虑积分 $\int_0^1 \frac{x^p \sin x}{1+x^q} dx$ 。由于

$$\begin{aligned} & \lim_{x \rightarrow +0} \left(x^{-1-p} \cdot \frac{x^p \sin x}{1+x^q} \right) \\ &= \lim_{x \rightarrow +0} \left(\frac{\sin x}{x} \cdot \frac{1}{1+x^q} \right) = 1, \end{aligned}$$

故积分 $\int_0^1 \frac{x^p \sin x}{1+x^q} dx$ 仅当 $-1-p < 1$ 即当 $p > -2$

时收敛，且是绝对收敛的。

再考虑积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 。(1) 若 $p \geq q$ ，则

对任何 $A > 1$ ，必存在正整数 N ，使 $2N\pi + \frac{\pi}{4} > A$

且当 $x \geq 2N\pi + \frac{\pi}{4}$ 时，恒有 $\frac{x^p}{1+x^q} > \frac{1}{3}$ 。于是，对

$A' = 2N\pi + \frac{\pi}{4}$, $A'' = 2N\pi + \frac{\pi}{2}$, 有

$$\begin{aligned} \left| \int_{A'}^{A''} \frac{x^p}{1+x^q} \sin x \, dx \right| &> \frac{1}{3} \int_{A'}^{A''} \sin x \, dx \\ &= \frac{\sqrt{2}}{6}, \end{aligned}$$

它不可能小于任给的 ε , 故积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} \, dx$ 发散. (2) 若 $p < q-1$, 取 $\alpha > 0$ 使 $p+\alpha < q-1$, 即 $q-p-\alpha > 1$, 由于

$$\begin{aligned} \lim_{x \rightarrow +\infty} x^{q-p-\alpha} \cdot \frac{x^p}{1+x^q} |\sin x| \\ = \lim_{x \rightarrow +\infty} \frac{x^q}{1+x^q} \cdot \frac{|\sin x|}{x^\alpha} = 0, \end{aligned}$$

故积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} \, dx$ 绝对收敛. (3) 现设 $q-1$

$\leq p < q$. 先证 $\int_1^{+\infty} \frac{x^p |\sin x|}{1+x^q} \, dx$ 发散. 事实上, 此

时, 可取 $A_0 > 1$, 使当 $x \geq A_0$ 时, $\frac{x^{p+1}}{1+x^q} \geq \frac{1}{3}$,

$$\begin{aligned} \text{故 } \int_{A_0}^{+\infty} \frac{x^p |\sin x|}{1+x^q} \, dx &= \int_{A_0}^{+\infty} \frac{x^{p+1}}{1+x^q} \cdot \left| \frac{\sin x}{x} \right| \, dx \\ &\geq \frac{1}{3} \int_{A_0}^{+\infty} \left| \frac{\sin x}{x} \right| \, dx = +\infty, \end{aligned}$$

从而 $\int_1^{+\infty} \frac{x^p |\sin x|}{1+x^q} \, dx$ 发散.

再证 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 收敛. 事实上, 若 $q = 0$,

则 $-1 \leq p < 0$, 此时积分 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$

$= \frac{1}{2} \int_1^{+\infty} x^p \sin x dx$ 显然收敛; 若 $q > 0$, 由于

$$\left(\frac{x^p}{1+x^q} \right)' = \frac{x^{p-1} [p - (q-p)x^q]}{(1+x^q)^2} < 0 \quad (\text{当 } x \text{ 充分}$$

大时),

故当 $x \rightarrow +\infty$ 时, $\frac{x^p}{1+x^q}$ 单调递减趋于零. 而

$\left| \int_1^A \sin x dx \right| = \left| \cos 1 - \cos A \right| \leq 2$ 有界, 故积分

$\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 收敛. 总之, 我们证明了: 当 $q-1$

$\leq p < q$ 时, $\int_1^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 条件收敛.

于是, 最后得结论: 积分 $\int_0^{+\infty} \frac{x^p \sin x}{1+x^q} dx$ 当

$p > -2$, $q > p+1$ 时绝对收敛; 当 $p > -2$, $p < q$
 $\leq p+1$ 时条件收敛.

2382. $\int_0^{+\infty} \frac{\sin \left(x + \frac{1}{x} \right)}{x^n} dx.$

解 当 $n \leq 0$ 时, 积分显然是发散的.

当 $n > 0$ 时, 首先考虑积分 $\int_a^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$

($a > 1$). 由于

$$\begin{aligned} & \int_a^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx \\ &= \int_a^{+\infty} \frac{\left(1 - \frac{1}{x^2}\right) \sin\left(x + \frac{1}{x}\right)}{x^n \left(1 - \frac{1}{x^2}\right)} dx, \end{aligned}$$

而

$$\begin{aligned} & \left| \int_a^A \left(1 - \frac{1}{x^2}\right) \sin\left(x + \frac{1}{x}\right) dx \right| \\ &= \left| \cos\left(a + \frac{1}{a}\right) - \cos\left(A + \frac{1}{A}\right) \right| \leq 2. \end{aligned}$$

又当 x 充分大时, 有

$$\frac{d}{dx} x^n \left(1 - \frac{1}{x^2}\right) = nx^{n-1} \left(x^2 - \frac{n-2}{n}\right) > 0,$$

故当 x 充分大时, 函数 $x^n \left(1 - \frac{1}{x^2}\right)$ 是增加的, 从而函

数 $\frac{1}{x^n \left(1 - \frac{1}{x^2}\right)}$ 当 $x \rightarrow +\infty$ 时递减趋于零. 由此可知,

积分 $\int_a^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$ 当 $n > 0$ 时收敛.

再考虑积分 $\int_0^{a'} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$ ($0 < a' < 1$).

设 $x = \frac{1}{t}$, 则

$$\int_0^{a'} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx = \int_{\frac{1}{a'}}^{+\infty} \frac{\sin\left(t + \frac{1}{t}\right)}{t^{2-n}} dt,$$

由前所述, 此积分仅当 $2 - n > 0$ 即当 $n < 2$ 时收敛.

请注意, $\int_{a'}^a \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$ ($0 < a' < 1 < a$)

是一个通常的积分, 它对任意 n 均有意义.

于是, 当 $0 < n < 2$ 时, 积分

$$\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$$

收敛.

可以证明: 积分

$$\int_0^{+\infty} \frac{\left|\sin\left(x + \frac{1}{x}\right)\right|}{x^n} dx$$

当 $0 < n < 2$ 时发散. 事实上,

$$\begin{aligned} \frac{\left|\sin\left(x + \frac{1}{x}\right)\right|}{x^n} &\geq \frac{\sin^2\left(x + \frac{1}{x}\right)}{x^n} \\ &= \frac{1 - \cos\left(2x + \frac{2}{x}\right)}{2x^n}, \end{aligned}$$

而当 $0 < n \leq 1$ 时, 积分 $\int_a^{+\infty} \frac{dx}{x^n}$ 显然发散, 积分

$\int_a^{+\infty} \frac{\cos\left(2x + \frac{2}{x}\right)}{x^n} dx$ 收敛 (仿前半段证明), 故当

$0 < n \leq 1$ 时, 积分 $\int_a^{+\infty} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$ 发散,

从而当 $0 < n \leq 1$ 时, 积分

$$\int_0^{+\infty} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$$

发散. 对于 $1 < n < 2$ 的情况, 可考虑对积分作变换

$x = \frac{1}{t}$, 则得

$$\begin{aligned} & \int_0^a \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx \\ &= \int_{\frac{1}{a}}^{+\infty} \frac{\left| \sin\left(t + \frac{1}{t}\right) \right|}{t^{2-n}} dt. \end{aligned}$$

仿前可知, 当 $0 < 2 - n \leq 1$ 即当 $1 \leq n < 2$ 时, 积分

$\int_0^a \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$ 发散. 从而, 当 $1 < n < 2$

时, 积分

$$\int_0^{+\infty} \frac{\left| \sin\left(x + \frac{1}{x}\right) \right|}{x^n} dx$$

发散

最后我们得到: 当 $0 < n < 2$ 时, 积分

$$\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^n} dx$$

条件收敛。

$$2383^+ \cdot \int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx,$$

式中 $P_m(x)$ 及 $P_n(x)$ 为整多项式, 且若 $x \geq a$, $P_n(x) > 0$ 。

解. 今仿2381题解之, 设

$$P_m(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m,$$

$$P_n(x) = b_0 x^n + b_1 x^{n-1} + \dots + b_n,$$

其中 m, n 是非负整数, $a_0 \neq 0, b_0 \neq 0$ 。

(1) 若 $n > m + 1$, 可取 $\alpha > 0$ 充分小, 使 $n - \alpha > m + 1$ 。由于

$$\begin{aligned} & \lim_{x \rightarrow +\infty} x^{n-m-\alpha} \cdot \left| \frac{P_m(x)}{P_n(x)} \sin x \right| \\ &= \lim_{x \rightarrow +\infty} \left| \frac{x^n P_m(x)}{x^m P_n(x)} \right| \cdot \frac{|\sin x|}{x^\alpha} = 0, \end{aligned}$$

而 $n - m - \alpha > 1$, 故积分 $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 绝对收敛。

(2) 若 $n = m + 1$ 。我们证明此时 $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$

条件收敛。事实上, 由于 $\lim_{x \rightarrow +\infty} \frac{x P_m(x)}{P_n(x)} = \frac{a_0}{b_0}$, 故存

在 $A_0 > a$, 使当 $x \geq A_0$ 时, 恒有 $\left| \frac{x P_m(x)}{P_n(x)} \right| > \frac{|a_0|}{2|b_0|}$,

于是

$$\begin{aligned} & \int_{A_0}^{+\infty} \left| \frac{P_m(x)}{P_n(x)} \sin x \right| dx \\ &= \int_{A_0}^{+\infty} \left| \frac{x P_m(x)}{P_n(x)} \right| \cdot \left| \frac{\sin x}{x} \right| dx \\ &\geq \frac{|a_0|}{2|b_0|} \int_{A_0}^{+\infty} \left| \frac{\sin x}{x} \right| dx = +\infty, \end{aligned}$$

故 $\int_a^{+\infty} \left| \frac{P_m(x)}{P_n(x)} \sin x \right| dx$ 发散。此外，易知 ($n=m+1$ 时)

$$\begin{aligned} \left(\frac{P_m(x)}{P_n(x)} \right)' &= \frac{1}{[P_n(x)]^2} \left\{ -a_0 b_0 x^{2m} \right. \\ &\quad \left. - 2a_1 b_0 x^{2m-1} + \dots + (a_{m-1} b_{m+1} \right. \\ &\quad \left. - a_m b_m) \right\}, \end{aligned}$$

故若 $a_0 b_0 > 0$ ，则当 x 充分大时， $\left(\frac{P_m(x)}{P_n(x)} \right)' < 0$ ，

函数 $\frac{P_m(x)}{P_n(x)}$ 减小；若 $a_0 b_0 < 0$ ，则当 x 充分大时，

$\left(\frac{P_m(x)}{P_n(x)} \right)' > 0$ ，函数 $\frac{P_m(x)}{P_n(x)}$ 增加。总之，当 $x \rightarrow$

$+\infty$ 时， $\frac{P_m(x)}{P_n(x)}$ 单调地趋于零。又显然可知

$\left| \int_a^A \sin x dx \right| \leq 2$ ，故积分 $\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x dx$ 收

敛。

(3) 若 $n < m + 1$, 由于 n, m 都是非负整数, 故 $n \leq m$, 因此

$$\lim_{x \rightarrow +\infty} \frac{P_m(x)}{P_n(x)} = \begin{cases} \frac{a_0}{b_0}, & \text{若 } n = m; \\ +\infty, & \text{若 } n < m \text{ 且 } a_0 b_0 > 0; \\ -\infty, & \text{若 } n < m \text{ 且 } a_0 b_0 < 0. \end{cases}$$

于是, 存在 $A^* > a$ 及 $\tau > 0$, 使当 $x \geq A^*$ 时 $\frac{P_m(x)}{P_n(x)}$

保持定号且 $\left| \frac{P_m(x)}{P_n(x)} \right| > \tau$. 今对任何 $A > a$, 可取正

整数 N , 使 $2N\pi + \frac{\pi}{4} \geq \max\{A, A^*\}$. 令 $A' = 2N\pi$

$+\frac{\pi}{4}$, $A'' = 2N\pi + \frac{\pi}{2}$, 则

$$\begin{aligned} \left| \int_{A'}^{A''} \frac{P_m(x)}{P_n(x)} \sin x \, dx \right| &> \tau \int_{A'}^{A''} \sin x \, dx \\ &= \frac{\tau\sqrt{2}}{2}, \end{aligned}$$

它不能小于任意的 ε ($0 < \varepsilon < \frac{\tau\sqrt{2}}{2}$), 故

$\int_a^{+\infty} \frac{P_m(x)}{P_n(x)} \sin x \, dx$ 发散.

最后, 我们得出: $\int_a^{+\infty} \frac{P_n(x)}{P_n(x)} \sin x \, dx$ 当 $n > m + 1$ 时绝对收敛; 当 $n = m + 1$ 时条件收敛.

2384. 若 $\int_a^{+\infty} f(x) \, dx$ 收敛, 则当 $x \rightarrow +\infty$ 时是否必有 $f(x) \rightarrow 0$?

研究例子:

$$(a) \int_0^{+\infty} \sin(x^2) dx, \quad (b) \int_0^{+\infty} (-1)^{[x^2]} dx.$$

解 不一定. 例如

(a) 积分 $\int_0^{+\infty} \sin(x^2) dx$ 收敛. 事实上, 它是 2380 题之特例: $p=0, q=2$. 但是, $\lim_{x \rightarrow +\infty} \sin(x^2)$ 不存在;

(b) 先证积分 $\int_0^{+\infty} (-1)^{[x^2]} dx$ 收敛. 事实上, 对任何 $A > 0$, 存在唯一的非负整数 n , 使 $\sqrt{n} \leq A < \sqrt{n+1}$. 显然 $A \rightarrow +\infty$ 相当于 $n \rightarrow \infty$. 当 $\sqrt{k} \leq x < \sqrt{k+1}$ (k —非负整数) 时, $[x^2] = k$. 于是

$$\begin{aligned} & \int_0^A (-1)^{[x^2]} dx \\ &= \sum_{k=0}^{n-1} \int_{\sqrt{k}}^{\sqrt{k+1}} (-1)^k dx + (-1)^n (A - \sqrt{n}) \\ &= \sum_{k=0}^{n-1} (-1)^k \frac{1}{\sqrt{k+1} + \sqrt{k}} + (-1)^n (A - \sqrt{n}). \end{aligned}$$

由于 $\frac{1}{\sqrt{k+1} + \sqrt{k}}$ 递减趋于 0 (当 $k \rightarrow \infty$ 时), 故

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (-1)^k \frac{1}{\sqrt{k+1} + \sqrt{k}} \text{ 存在有限 (参看 2656 题}$$

前面的变号级数的莱布尼兹判别法), 设为 S . 又显然

$$|(-1)^n (A - \sqrt{n})| < \sqrt{n+1} - \sqrt{n}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0 \quad (\text{当 } n \rightarrow \infty \text{ 时}),$$

故 $\lim_{A \rightarrow +\infty} \int_0^A (-1)^{[x^2]} dx = S$, 因此

积分 $\int_0^{+\infty} (-1)^{[x^2]} dx$ 收敛.

但显然 $\lim_{x \rightarrow +\infty} (-1)^{[x^2]}$ 不存在.

2385. 于 $[a, b]$ 上有定义的, 无界函数 $f(x)$ 可否把函数 $f(x)$ 的收敛广义积分

$$\int_a^b f(x) dx$$

看作对应的积分和式

$$\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

的极限? 式中 $x_i \leq \xi_i \leq x_{i+1}$ 及 $\Delta x_i = x_{i+1} - x_i$.

解 不能. 因为若 $c (a \leq c \leq b)$ 是瑕点, 则对于 $[a, b]$ 的任何分法, 不论其 $\max |\Delta x_i|$ 多么小, 当分法确定以后, 设 $c \in [x_i, x_{i+1}]$, 则总可以取 ξ_i , 使

$\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$ 大于任何预先给定的值. 因此, 当

$\max |\Delta x_i| \rightarrow 0$ 时, $\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$ 不可能具有有限极

限.

2386. 设:

$$\int_a^{+\infty} f(x) dx \quad (1)$$

收敛，函数 $\varphi(x)$ 有界，则积分

$$\int_a^{+\infty} f(x)\varphi(x)dx \quad (2)$$

是否必定收敛？举出适当的例子。

若积分(1)绝对收敛，问积分(2)的收敛性如何？

解 不。例如，积分

$$\int_0^{+\infty} \frac{\sin x}{x} dx$$

收敛*)。且 $\varphi(x) = \sin x$ 有界，但是积分

$$\int_0^{+\infty} \frac{\sin^2 x}{x} dx$$

是发散的**)

若积分(1)绝对收敛， $\varphi(x)$ 有界，则积分(2)一定是绝对收敛的。事实上，设 $|\varphi(x)| \leq L$ ，则由不等式

$$|f(x)\varphi(x)| \leq L \cdot |f(x)|$$

及 $\int_a^{+\infty} |f(x)| dx$ 的收敛性即可获证。

*) 利用2378题的结果。

***) 利用2368题的结果。

2387. 证明，若 $\int_a^{+\infty} f(x) dx$ 收敛， $f(x)$ 为单调函数，则

$$f(x) = o\left(\frac{1}{x}\right) \quad *)$$

证 不妨设 $f(x)$ 单调减小。先证当 $x \geq a$ 时, $f(x) \geq 0$ 。若不然, 则存在点 $c \geq a$, 使 $f(c) < 0$ 。由于 $f(x)$ 单调减小, 故当 $x \geq c$ 时, $f(x) \leq f(c)$, 从而

$$\int_c^{+\infty} f(x) dx \leq \int_c^{+\infty} f(c) dx = -\infty.$$

因此, 积分

$$\int_a^{+\infty} f(x) dx$$

发散, 这与积分 $\int_a^{+\infty} f(x) dx$ 收敛矛盾。于是, $f(x)$ 为非负的单调函数。

下面证明 $f(x) = o\left(\frac{1}{x}\right)$ 。由于积分

$$\int_a^{+\infty} f(x) dx$$

收敛, 故对任给的 $\varepsilon > 0$, 总存在 $A > a$, 使当 $x > A$ 时, 恒有

$$\left| \int_{\frac{x}{2}}^x f(t) dt \right| < \frac{\varepsilon}{2}.$$

但是

$$\begin{aligned} \left| \int_{\frac{x}{2}}^x f(t) dt \right| &= \int_{\frac{x}{2}}^x f(t) dt \geq f(x) \cdot \left(x - \frac{x}{2}\right) \\ &= \frac{x}{2} f(x), \end{aligned}$$

故当 $x > A$ 时,

$$0 \leq x f(x) < \varepsilon,$$

即

$$\lim_{x \rightarrow +\infty} xf(x) = 0 \quad \text{或} \quad f(x) = o\left(\frac{1}{x}\right).$$

如果 $f(x)$ 单调增大, 则可考虑 $-f(x)$ (它是单调减小的), 同法可证得 $f(x) = o\left(\frac{1}{x}\right)$.

*) 原题为 $f(x) = O\left(\frac{1}{x}\right)$, 现在的结果更好.

2388. 设函数 $f(x)$ 于区间 $0 < x \leq 1$ 内是单调的函数, 且在点 $x = 0$ 的邻域内是无界的, 证明若

$$\int_0^1 f(x) dx$$

存在, 则

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

证 设函数 $f(x)$ 在 $(0, 1]$ 上是单调下降的. 这时 $\lim_{x \rightarrow +0} f(x) = +\infty$. 先设 $f(x) \geq 0$ ($0 < x \leq 1$ 时).

由于积分

$$\int_0^1 f(x) dx$$

存在, 故把区间 $[0, 1]$ n 等分后, 即得

$$\begin{aligned} \int_0^1 f(x) dx &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) dx \\ &< \int_0^{\frac{1}{n}} f(x) dx + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} \end{aligned}$$

$$< \int_0^1 f(x) dx + \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n}.$$

另一方面，又有

$$\int_0^1 f(x) dx > \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n}.$$

从而就有

$$0 < \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) < \int_0^1 f(x) dx.$$

由于 $\lim_{n \rightarrow \infty} \int_0^1 f(x) dx = 0$ ，故

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

如果不满足 $f(x) \geq 0$ ，即 $f(x)$ 可正可负，则函数 $\varphi(x) = f(x) - f(1)$ 满足 $\varphi(x) \geq 0$ ($0 < x \leq 1$)，且同样是单调下降， $\lim_{x \rightarrow +0} \varphi(x) = +\infty$ 。故根据已证的结果，知

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi\left(\frac{k}{n}\right) = \int_0^1 \varphi(x) dx,$$

即

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[f\left(\frac{k}{n}\right) - f(1) \right] \\ &= \int_0^1 \left[f(x) - f(1) \right] dx, \end{aligned}$$

由此即得

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

当 $f(x)$ 在 $[0, 1]$ 单调增加时 (这时 $\lim_{x \rightarrow +0} f(x) = -\infty$)，只需对函数 $-f(x)$ 应用上述结果即获证。

2389. 证明：若函数 $f(x)$ 于区间 $0 < x < a$ 内是单调的，且

$$\int_0^a x^p f(x) dx$$

存在，则

$$\lim_{x \rightarrow +0} x^{p+1} f(x) = 0.$$

证 不妨设 $f(x)$ 在 $0 < x < a$ 是单调递减的。先设存在 $0 < \delta < a$ 使在 $0 < x < \delta$ 时 $f(x) \geq 0$ 。这时，当 $0 < x < \delta$ 时，有

$$\begin{aligned} \int_{\frac{x}{2}}^x t^p f(t) dt &\geq f(x) \int_{\frac{x}{2}}^x t^p dt \\ &= C_p x^{p+1} f(x) \geq 0, \end{aligned}$$

$$\text{其中 } C_p = \begin{cases} \frac{1 - \left(\frac{1}{2}\right)^{p+1}}{p+1}, & \text{当 } p \neq -1 \text{ 时,} \\ \ln 2, & \text{当 } p = -1 \text{ 时.} \end{cases}$$

故 C_p 是正的常数。

于是，由 $\int_0^a x^p f(x) dx$ 存在，知

$$\lim_{x \rightarrow +0} \int_{\frac{x}{2}}^x t^p f(t) dt = 0, \text{ 从而 } \lim_{x \rightarrow +0} x^{p+1} f(x) = 0.$$

再设不存在上述 δ 。于是，根据 $f(x)$ 的递减性，

知当 $0 < x < a$ 时恒有 $f(x) < 0$ 。于是，当 $0 < x < \frac{a}{2}$

时, 有

$$\int_x^{2x} t^p f(t) dt \leq f(x) \int_x^{2x} t^p dt \\ = C_p^* x^{p+1} f(x) \leq 0,$$

$$\text{其中 } C_p^* = \begin{cases} \frac{2^{p+1} - 1}{p+1}, & \text{当 } p \neq -1 \text{ 时;} \\ \ln 2, & \text{当 } p = -1 \text{ 时.} \end{cases}$$

故 C_p^* 也是正的常数.

$$\text{于是, } |x^{p+1} f(x)| \leq \frac{1}{C_p^*} \left| \int_x^{2x} t^p f(t) dt \right|.$$

根据 $\int_0^2 x^p f(x) dx$ 的存在性, 知

$$\lim_{x \rightarrow 0} \int_x^{2x} t^p f(t) dt = 0,$$

从而 $\lim_{x \rightarrow 0} x^{p+1} f(x) = 0$. 证完.

2390. 证明

$$(a) \quad V.P. \int_{-1}^1 \frac{dx}{x} = 0,$$

$$(b) \quad V.P. \int_0^{+\infty} \frac{dx}{1-x^2} = 0;$$

$$(B) \quad V.P. \int_{-\infty}^{+\infty} \sin x dx = 0.$$

证 (a) 由于

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{-1}^{0-\epsilon} \frac{dx}{x} + \int_{0+\epsilon}^1 \frac{dx}{x} \right\}$$

$$= \lim_{\varepsilon \rightarrow +0} (\ln \varepsilon - \ln 1 + \ln 1 - \ln \varepsilon) = 0,$$

所以

$$V.P. \int_{-1}^1 \frac{dx}{x} = 0;$$

(6) 由于

$$\begin{aligned} & \lim_{\substack{\varepsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left(\int_0^{1-\varepsilon} \frac{dx}{1-x^2} + \int_{1+\varepsilon}^b \frac{dx}{1-x^2} \right) \\ &= \lim_{\substack{\varepsilon \rightarrow +0 \\ b \rightarrow +\infty}} \left(\frac{1}{2} \ln \left| \frac{2-\varepsilon}{\varepsilon} \right| + \frac{1}{2} \ln \left| \frac{1+b}{1-b} \right| \right. \\ & \quad \left. - \frac{1}{2} \ln \left| \frac{2+\varepsilon}{\varepsilon} \right| \right) \\ &= \frac{1}{2} \lim_{\varepsilon \rightarrow +0} \ln \left| \frac{2-\varepsilon}{2+\varepsilon} \right| = 0, \end{aligned}$$

所以

$$V.P. \int_0^{+\infty} \frac{dx}{1-x^2} = 0;$$

(B) 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_{-b}^b \sin x \, dx = \lim_{b \rightarrow +\infty} (-\cos b + \cos b) \\ &= 0, \end{aligned}$$

所以

$$V.P. \int_{-\infty}^{+\infty} \sin x \, dx = 0.$$

2391. 证明: 当 $x \geq 0$ 且 $x \neq 1$ 时,

$$\operatorname{li} x = V.P. \int_0^x \frac{d\xi}{\ln \xi}$$

存在*)。

证 当 $0 \leq x < 1$ 时, 由于 $\lim_{\xi \rightarrow +0} \frac{1}{\ln \xi} = 0$, 故将 $\frac{1}{\ln x}$ 在 $x = 0$ 处补充定义后成为连续函数, 于是积分存在。

当 $x > 1$ 时, 首先注意到下面这样一个结论: 当 $a < c < b$ 时,

$$\begin{aligned} V.P. \int_a^b \frac{dx}{x-c} &= \lim_{\epsilon \rightarrow +0} \left(\int_a^{c-\epsilon} \frac{dx}{x-c} + \int_{c+\epsilon}^b \frac{dx}{x-c} \right) \\ &= \ln \frac{b-c}{c-a}. \end{aligned}$$

其次, 利用具比亚诺型余项的台劳公式, 有

$$\ln x = (x-1) + [\alpha(x)-1] \frac{(x-1)^2}{2},$$

式中 $\lim_{x \rightarrow 1} \alpha(x) = 0$ 。由此即得

$$\frac{1}{\ln x} = \frac{1}{x-1} - \frac{\frac{1}{2}[\alpha(x)-1]}{1 + \frac{[\alpha(x)-1](x-1)}{2}},$$

上述等式右端的第二项在 $x = 1$ 的附近保持有界, 且对于任意的 x 值连续, 因而是可积分的。第一项的“主值”如前所述, 它是存在的。

于是, 当 $x \geq 0$ 且 $x \neq 1$ 时, $\ln x$ 存在。

*) 原题误为“当 $x \geq 0$ 时, ...”。

求下列积分:

$$2392. \text{ V.P. } \int_0^{+\infty} \frac{dx}{x^2 - 3x + 2}.$$

解 由于

$$\begin{aligned} & \lim_{\substack{\varepsilon \rightarrow +0 \\ \eta \rightarrow +0 \\ b \rightarrow +\infty}} \left(\int_0^{1-\varepsilon} \frac{dx}{x^2 - 3x + 2} + \int_{1+\varepsilon}^{2-\eta} \frac{dx}{x^2 - 3x + 2} \right. \\ & \quad \left. + \int_{2+\eta}^b \frac{dx}{x^2 - 3x + 2} \right) \\ &= \lim_{\substack{\varepsilon \rightarrow +0 \\ \eta \rightarrow +0 \\ b \rightarrow +\infty}} \left(\ln \frac{\varepsilon+1}{\varepsilon} - \ln 2 + \ln \frac{\eta}{1-\eta} - \ln \frac{1-\varepsilon}{\varepsilon} \right. \\ & \quad \left. + \ln \left| \frac{b-2}{b-1} \right| - \ln \frac{\eta}{1+\eta} \right) \\ &= \lim_{\substack{\varepsilon \rightarrow +0 \\ \eta \rightarrow +0}} \left(\ln \frac{\varepsilon+1}{1-\varepsilon} - \ln 2 + \ln \frac{1+\eta}{1-\eta} \right) \\ &= -\ln 2 = \ln \frac{1}{2}. \end{aligned}$$

所以

$$\text{V.P. } \int_0^{+\infty} \frac{dx}{x^2 - 3x + 2} = \ln \frac{1}{2}.$$

$$2393. \text{ V.P. } \int_{\frac{1}{2}}^2 \frac{dx}{x \ln x}.$$

解 由于

$$\lim_{\varepsilon \rightarrow +0} \left[\int_{\frac{1}{2}}^{1-\varepsilon} \frac{dx}{x \ln x} + \int_{1+\varepsilon}^2 \frac{dx}{x \ln x} \right]$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow +0} (\ln |\ln(1-\varepsilon)| - \ln(\ln 2) + \ln(\ln 2) \\
&\quad - \ln |\ln(1+\varepsilon)|) \\
&= \lim_{\varepsilon \rightarrow +0} \ln \left| \frac{\ln(1-\varepsilon)}{\ln(1+\varepsilon)} \right| = \ln \left| \lim_{\varepsilon \rightarrow +0} \frac{\ln(1-\varepsilon)}{\ln(1+\varepsilon)} \right| \\
&= \ln \left| \lim_{\varepsilon \rightarrow +0} \frac{\frac{-1}{1-\varepsilon}}{\frac{1}{1+\varepsilon}} \right| = \ln 1 = 0,
\end{aligned}$$

所以

$$V.P. \int_{\frac{1}{2}}^2 \frac{dx}{x \ln x} = 0.$$

$$2394. \quad V.P. \int_{-\infty}^{+\infty} \frac{1+x}{1+x^2} dx.$$

解 由于

$$\begin{aligned}
&\lim_{b \rightarrow +\infty} \int_{-b}^b \frac{1+x}{1+x^2} dx \\
&= \lim_{b \rightarrow +\infty} \left[\operatorname{arc} \operatorname{tg} b - \operatorname{arc} \operatorname{tg} (-b) + \frac{1}{2} \ln(1+b^2) \right. \\
&\quad \left. - \frac{1}{2} \ln(1+b^2) \right] \\
&= 2 \lim_{b \rightarrow +\infty} \operatorname{arc} \operatorname{tg} b = \pi,
\end{aligned}$$

所以

$$V.P. \int_{-\infty}^{+\infty} \frac{1+x}{1+x^2} dx = \pi.$$

$$2395. \quad V.P. \int_{-\infty}^{+\infty} \operatorname{arc} \operatorname{tg} x dx.$$

解 由于

$$\begin{aligned} & \lim_{b \rightarrow +\infty} \int_{-b}^b \operatorname{arc} \operatorname{tg} x \, dx \\ &= \lim_{b \rightarrow +\infty} \left[b \operatorname{arc} \operatorname{tg} b - (-b) \operatorname{arc} \operatorname{tg}(-b) \right. \\ & \quad \left. - \frac{1}{2} \ln(1+b^2) + \frac{1}{2} \ln(1+b^2) \right] = 0, \end{aligned}$$

所以

$$V.P. \int_{-\infty}^{+\infty} \operatorname{arc} \operatorname{tg} x \, dx = 0.$$

§ 5 . 面积的计算法

1° 直角坐标系中的面积 由两条连续的曲线 $y = y_1(x)$ 和 $y = y_2(x)$ [$y_2(x) \geq y_1(x)$] 与 Ox 轴的两条垂线 $x = a$ 和 $x = b$ 所围成的面积 $S = A_1 A_2 B_2 B_1$ (图 4.14) 等于

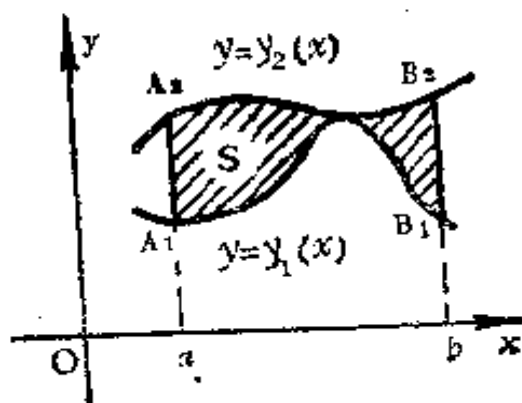


图 4.14

$$S = \int_a^b [y_2(x) - y_1(x)] dx.$$

2° 参数形状表出的曲线所围成的面积 若 $x = x(t)$, $y = y(t)$ [$0 \leq t \leq T$] 为一逐段平滑的简单封闭曲线 C 的参数方程式, 面积 S 表由此曲线所围在它左侧的面积 (图

4.15), 则

$$S = - \int_0^T y(t)x'(t)dt = \int_0^T x(t)y'(t)dt$$

或

$$S = \frac{1}{2} \int_0^T [x(t)y'(t) - x'(t)y(t)]dt.$$

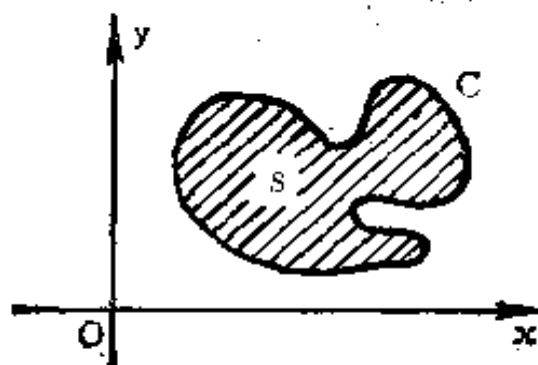


图 4.15

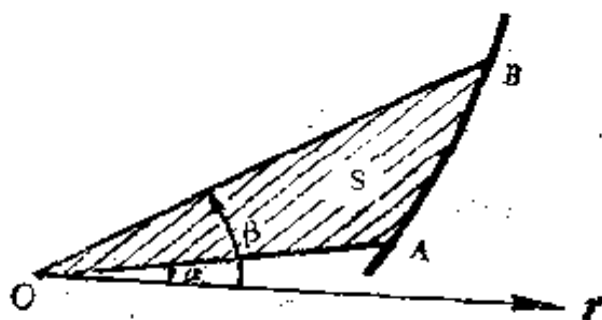


图 4.16

3° 极坐标系中的面积 由连续的曲线 $r = r(\varphi)$ 和两条半射线 $\varphi = \alpha$ 和 $\varphi = \beta$ 所围成的面积 $S = OAB$ (图4.16) 等于

$$S = \frac{1}{2} \int_{\alpha}^{\beta} r^2(\varphi) d\varphi.$$

2396. 证明 正抛物线拱的面积等于

$$S = \frac{2}{3}bh,$$

式中 b 为底, h 为拱的高 (图4.17)。

证 设抛物线的方程为

$$y = Ax^2 + Bx + C,$$

则当 $x = \pm \frac{b}{2}$ 时, 得

$$y = \frac{Ab^2}{4} \pm \frac{Bb}{2} + C = 0;$$

当 $x = 0$ 时, 得

$$y = C = h.$$

解之得

$$A = -\frac{4h}{b^2}, \quad B = 0.$$

从而

$$y = -\frac{4h}{b^2}x^2 + h.$$

于是, 所求的面积为

$$\begin{aligned} S &= 2 \int_0^{\frac{b}{2}} \left(h - \frac{4h}{b^2}x^2 \right) dx \\ &= 2 \left(hx - \frac{4h}{3b^2}x^3 \right) \Big|_0^{\frac{b}{2}} = \frac{2}{3}bh. \end{aligned}$$

求下列直角坐标方程所表曲线围成的面积*).

2397. $ax = y^2, \quad ay = x^2.$

解 如图4.18所示, 交点为

$$A(a, a) \text{ 及 } O(0, 0).$$

所求的面积为

$$S = \int_0^a \left(\sqrt{ax} - \frac{x^2}{a} \right) dx$$

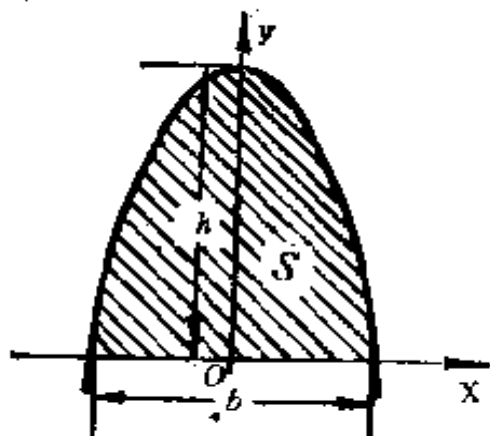


图 4.17.

*) 在第四章的这一节和以后各节都把一切参数当作是正的.

$$= \left[\frac{2}{3a} - (ax)^{\frac{3}{2}} - \frac{1}{3a} x^3 \right] \Big|_0^a = \frac{a^2}{3}.$$

2398. $y = x^2$, $x + y = 2$.

解 如图4.19所示, 交点为

$A(-2, 4)$ 及 $B(1, 1)$.

所求的面积为

$$\begin{aligned} S &= \int_{-2}^1 [(2-x) - x^2] dx \\ &= \left(2x - \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_{-2}^1 = 4\frac{1}{2}. \end{aligned}$$

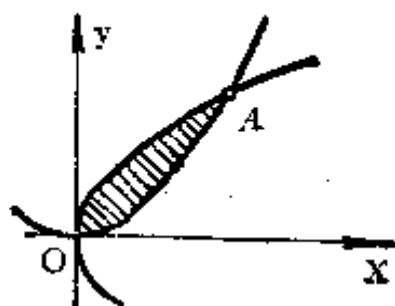


图 4.18

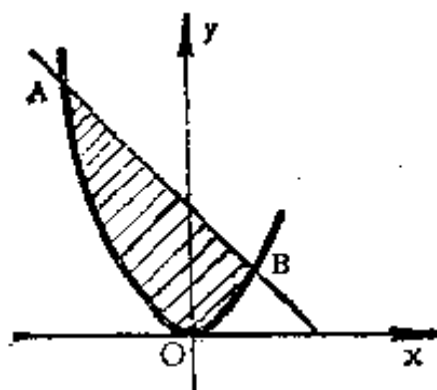


图 4.19

2399. $y = 2x - x^2$, $x + y = 0$.

解 如图4.20所示, 交点为

$A(3, -3)$ 及 $O(0, 0)$.

所求的面积为

$$\begin{aligned} S &= \int_0^3 [(2x - x^2) - (-x)] dx \\ &= \left(\frac{3x^2}{2} - \frac{1}{3}x^3 \right) \Big|_0^3 = 4\frac{1}{2}. \end{aligned}$$

2400. $y = |\lg x|$; $y = 0$, $x = 0.1$, $x = 10$.

解 如图4.21所示, 所求的面积为

$$\begin{aligned} S &= - \int_{0.1}^1 \lg x dx + \int_1^{10} \lg x dx \\ &= (-x \lg x + x \lg e) \Big|_{0.1}^1 \\ &\quad + (x \lg x - x \lg e) \Big|_1^{10} \\ &= 9.9 - 8.1 \lg e \doteq 6.38. \end{aligned}$$

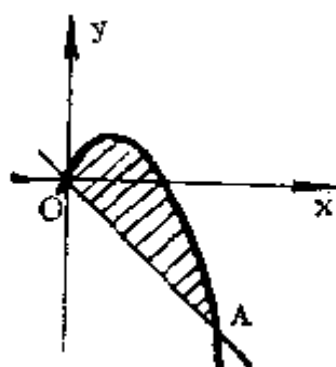


图 4.20

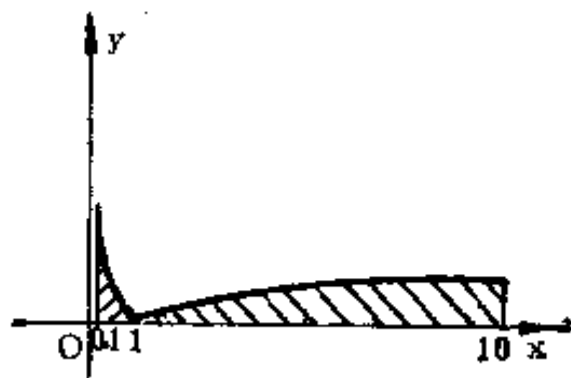


图 4.21

2401. $y = x$, $y = x + \sin^2 x$ ($0 \leq x \leq \pi$).

解 所求的面积为

$$\begin{aligned} S &= \int_0^{\pi} (x + \sin^2 x - x) dx \\ &= \left(\frac{x}{2} - \frac{1}{4} \sin 2x \right) \Big|_0^{\pi} = \frac{\pi}{2}. \end{aligned}$$

2402. $y = \frac{a^3}{a^2 + x^2}$, $y = 0$.

解 所求的面积为

$$\begin{aligned} S &= \int_{-\infty}^{+\infty} \frac{a^3}{a^2+x^2} dx = 2a^3 \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{a^2+x^2} \\ &= 2a^3 \cdot \lim_{b \rightarrow +\infty} \frac{1}{a} \operatorname{arc} \operatorname{tg} b = \pi a^2. \end{aligned}$$

$$2403. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

解 所求的面积为

$$\begin{aligned} S &= 4 \int_0^a \frac{b}{a} \sqrt{a^2-x^2} dx \\ &= \frac{4b}{a} \left(\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \operatorname{arc} \sin \frac{x}{a} \right) \Big|_0^a \\ &= \pi ab. \end{aligned}$$

$$2404. y^2 = x^2(a^2-x^2).$$

解 如图4.22所示, 图形对称于原点.

所求的面积为

$$\begin{aligned} S &= 4 \int_0^a x \sqrt{a^2-x^2} dx = -\frac{4}{3} (a^2-x^2)^{\frac{3}{2}} \Big|_0^a \\ &= \frac{4}{3} a^3. \end{aligned}$$

$$2405. y^2 = 2px, \quad 27py^2 = 8(x-p)^3.$$

解 曲线 $L_1: 27py^2 = 8(x-p)^3$ 与曲线 $L_2: y^2 = 2px$ 在第一象限内的交点为

$$A(4p, 2\sqrt{2}p),$$

如图4.23所示. 所求的面积为

$$\begin{aligned}
 S &= 2 \int_0^{2\sqrt{2}p} \left[\left(p + \frac{3}{2} p^{\frac{1}{2}} y^{\frac{2}{3}} \right) - \frac{1}{2p} y^2 \right] dy \\
 &= 2 \left(py + \frac{9}{10} p^{\frac{1}{2}} y^{\frac{5}{3}} - \frac{1}{6p} y^3 \right) \Big|_0^{2\sqrt{2}p} \\
 &= \frac{88}{15} \sqrt{2} p^2.
 \end{aligned}$$

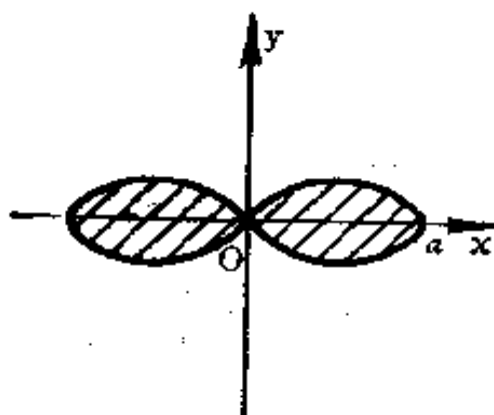


图 4.22

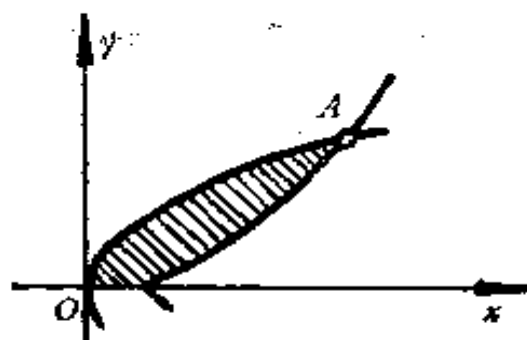


图 4.23

2406. $Ax^2 + 2Bxy + Cy^2 = 1$ ($AC - B^2 > 0$).

解 解此方程, 得

$$y_1 = \frac{-Bx - \sqrt{B^2x^2 - C(Ax^2 - 1)}}{C},$$

及

$$y_2 = \frac{-Bx + \sqrt{B^2x^2 - C(Ax^2 - 1)}}{C}.$$

当 $B^2x^2 - C(Ax^2 - 1) \geq 0$, 即 $|x| \leq \sqrt{\frac{C}{AC - B^2}}$ 时,

y_1 及 y_2 才有实数值.

设

$$a = \sqrt{\frac{C}{AC - B^2}},$$

则所求的面积为

$$\begin{aligned} S &= \int_{-a}^a (y_2 - y_1) dx \\ &= \frac{2}{C} \int_{-a}^a \sqrt{C^2 - (AC - B^2)x^2} dx \\ &= \frac{2}{C} \sqrt{AC - B^2} \int_{-a}^a \sqrt{a^2 - x^2} dx \\ &= \frac{2}{C} \sqrt{AC - B^2} \cdot \frac{\pi}{2} a^2 = \frac{\pi}{\sqrt{AC - B^2}} \end{aligned}$$

2407. $y^2 = \frac{x^3}{2a-x}$ (蔓叶线), $x=2a$.

解 所求的面积为

$$\begin{aligned} S &= 2 \int_0^{2a} x \sqrt{\frac{x}{2a-x}} dx \\ &= 16a^2 \int_0^{+\infty} \frac{t^4}{(t^2+1)^3} dt^{**}) \\ &= 16a^2 \lim_{t \rightarrow +\infty} \int_0^t \left[\frac{1}{t^2+1} - \frac{2}{(t^2+1)^2} \right. \\ &\quad \left. + \frac{1}{(t^2+1)^3} \right] dt \\ &= 16a^2 \lim_{t \rightarrow +\infty} \left\{ \frac{3}{8} \operatorname{arc} \operatorname{tg} t - \frac{5t}{8(t^2+1)} \right. \\ &\quad \left. + \frac{t}{4(t^2+1)^2} \right\}^{***)} \Big|_0^t \end{aligned}$$

$$= 3\pi a^2.$$

*) 设 $t = \sqrt{\frac{x}{2a-x}}$.

**） 利用1921题的递推公式.

2408. $x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y}$
 $-\sqrt{a^2 - y^2}$ (曳物线),
 $y = 0$.

解 如图4.24所示, 所求的面积为

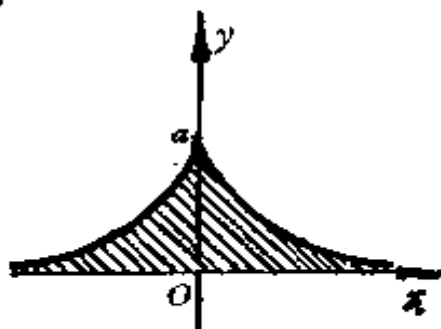


图 4.24

$$\begin{aligned} S &= 2 \int_0^a \left(a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \right) dy \\ &= 2a \lim_{\epsilon \rightarrow +0} \int_{\epsilon}^a \ln \frac{a + \sqrt{a^2 - y^2}}{y} dy - \\ &\quad - 2 \left(\frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \arcsin \frac{y}{a} \right) \Big|_0^a \\ &= 2a \lim_{\epsilon \rightarrow +0} \left(y \ln \frac{a + \sqrt{a^2 - y^2}}{y} \right. \\ &\quad \left. + a \arcsin \frac{y}{a} \right) \Big|_{\epsilon}^a - \frac{\pi a^2}{2} \\ &= \pi a^2 - \frac{\pi a^2}{2} = \frac{\pi a^2}{2}. \end{aligned}$$

2409. $y^2 = \frac{x^n}{(1+x^{n+2})^2}$ ($x > 0$; $n > -2$).

解 所求的面积为

$$\begin{aligned}
S &= 2 \int_0^{+\infty} \frac{x^{\frac{n}{2}}}{1+x^{n+2}} dx \\
&= 2 \lim_{\substack{\varepsilon \rightarrow +0 \\ b \rightarrow +\infty}} \int_{\varepsilon}^b \frac{2}{n+2} \cdot \frac{dt}{1+t^2} \quad *) \\
&= 2 \cdot \frac{2}{n+2} \lim_{b \rightarrow +\infty} \arctg t \Big|_0^b = \frac{2\pi}{n+2}.
\end{aligned}$$

*) 设 $t = x^{\frac{n+2}{2}}$.

2410. $y = e^{-x} \sin x$, $y = 0$ ($x \geq 0$).

解 令 $\sin x = 0$, 得 $x = k\pi$ ($k = 0, \pm 1, \pm 2, \dots$).
 当 $x \geq 0$ 时, 由于 $\sin x$ 在 $(\pi, 2\pi), (3\pi, 4\pi), \dots,$
 $((2k+1)\pi, 2k\pi), \dots$ 中的值为负, 而在 $(0, \pi),$
 $(2\pi, 3\pi), \dots, (2k\pi, (2k+1)\pi), \dots$ 中的值为正,
 故所求的面积为

$$\begin{aligned}
S &= \int_0^{\pi} e^{-x} \sin x dx - \int_{\pi}^{2\pi} e^{-x} \sin x dx \\
&\quad + \int_{2\pi}^{3\pi} e^{-x} \sin x dx - \dots + \\
&\quad + (-1)^k \int_{k\pi}^{(k+1)\pi} e^{-x} \sin x dx + \dots \\
&= \lim_{n \rightarrow +\infty} \sum_{k=0}^n (-1)^k \int_{k\pi}^{(k+1)\pi} e^{-x} \sin x dx \\
&= \lim_{n \rightarrow +\infty} \sum_{k=0}^n (-1)^k \cdot \frac{-e^{-x}(\sin x + \cos x)}{2} \Big|_{k\pi}^{(k+1)\pi} \\
&= \lim_{n \rightarrow +\infty} \sum_{k=0}^n (-1)^{k+1} \cdot \frac{1}{2} \left[e^{-(k+1)\pi} \cos(k+1)\pi \right.
\end{aligned}$$

$$\begin{aligned}
& -e^{-kx} \cos k\pi \Big] \\
& = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{2} \cdot (-1)^{k+1} \{ (-1)^{k+1} e^{-(k+1)x} \\
& \quad - (-1)^k e^{-kx} \} \\
& = \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{k=0}^n \left[e^{-(k+1)x} + e^{-kx} \right] \\
& = \frac{1}{2} \lim_{n \rightarrow \infty} \left\{ 1 + 2e^{-x} \sum_{k=0}^{n-1} e^{-kx} + e^{-(n+1)x} \right\} \\
& = \frac{1}{2} \lim_{n \rightarrow \infty} \left\{ 1 + 2e^{-x} \cdot \frac{1 - e^{-nx}}{1 - e^{-x}} + e^{-(n+1)x} \right\} \\
& = \frac{1}{2} \left(1 + \frac{2e^{-x}}{1 - e^{-x}} \right) = \frac{1}{2} \cdot \frac{e^x + 1}{e^x - 1} \\
& = \frac{1}{2} \operatorname{cth} \frac{\pi}{2} = 0.545.
\end{aligned}$$

2411. 抛物线 $y^2 = 2x$ 分圆 $x^2 + y^2 = 8$ 的面积为两部分, 这两部分的比如何?

解 抛物线 $y^2 = 2px$ 和圆 $x^2 + y^2 = 8$ 在第一象限内的交点为 $A(2, 2)$.

设这两部分的面积分别为 S_1 及 S_2 (图4.25), 则有

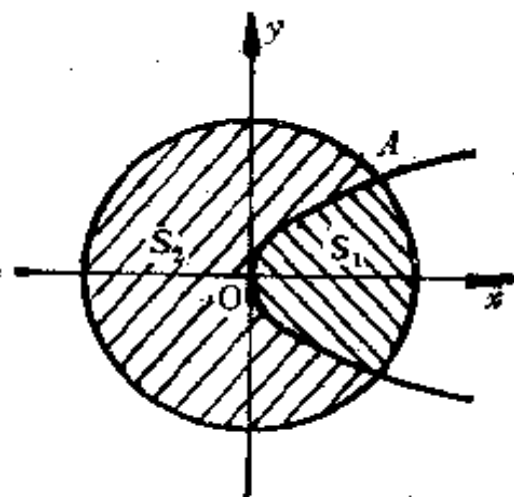


图 4.25

$$\begin{aligned}
 S_1 &= 2 \int_0^2 \left(\sqrt{8-y^2} - \frac{y^2}{2} \right) dy \\
 &= 2 \left(\frac{y}{2} \sqrt{8-y^2} + \frac{8}{2} \arcsin \frac{y}{2\sqrt{2}} - \frac{1}{6} y^3 \right) \Big|_0^2 = 2\pi + \frac{4}{3},
 \end{aligned}$$

及

$$S_2 = 8\pi - \left(2\pi + \frac{4}{3} \right) = 6\pi - \frac{4}{3}.$$

于是，它们之比为

$$\frac{S_1}{S_2} = \frac{2\pi + \frac{4}{3}}{6\pi - \frac{4}{3}} = \frac{3\pi + 2}{9\pi - 2}.$$

2412. 把双曲线 $x^2 - y^2 = a$ 上的点 $M(x, y)$ 的坐标表成为双曲线扇形 $S = OM'M$ 面积的函数。这个扇形是由双曲线的弧 $M'M$ 与二射线 OM 及 OM' 所围成，其中 $M'(x, -y)$ 是对于 Ox 轴与 M 对称的点。

解 如图4.26所示，则有

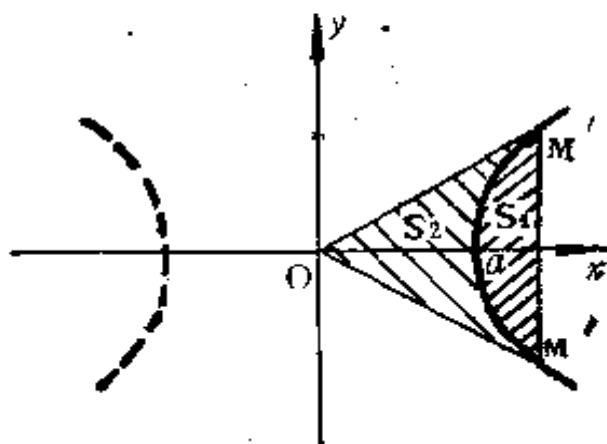


图 4.26

$$\begin{aligned} \frac{S_1}{2} &= \int_a^x \sqrt{x^2 - a^2} dx \\ &= \left[\frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2 - a^2}) \right]_a^x \\ &= \frac{1}{2} xy - \frac{a^2}{2} \ln \frac{x+y}{a} \end{aligned}$$

及

$$S_2 = 2 \left(\frac{xy}{2} - \frac{S_1}{2} \right) = a^2 \ln \frac{x+y}{a}$$

若记 $S_2 = S$, 则由上式得

$$x + y = a e^{\frac{S}{a^2}} \quad (1)$$

以 (1) 式代入 $x^2 - y^2 = a^2$ 中, 易得

$$x - y = a e^{-\frac{S}{a^2}} \quad (2)$$

由 (1) 式及 (2) 式, 解得

$$x = a \cdot \frac{e^{\frac{S}{a^2}} + e^{-\frac{S}{a^2}}}{2} = a \operatorname{ch} \frac{S}{a^2}$$

及

$$y = a \cdot \frac{e^{\frac{S}{a^2}} - e^{-\frac{S}{a^2}}}{2} = a \operatorname{sh} \frac{S}{a^2}$$

求由下列参数方程式所表曲线围成的面积:

2413. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$) (摆线) 及 $y = 0$.

解 所求的面积为

$$\begin{aligned}
 S &= \int_0^{2\pi} a(1 - \cos t) \cdot a(1 - \cos t) dt \\
 &= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2} \right) dt \\
 &= a^2 \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t \right) \Big|_0^{2\pi} \\
 &= 3\pi a^2.
 \end{aligned}$$

由此可见，所求摆线一拱的面积等于原来母圆面积的三倍。

2414. $x = 2t - t^2, y = 2t^2 - t^3.$

解 当 $t = 0$ 及 2 时, $x = 0, y = 0,$

当 $0 < t < 2$ 时,

$$x > 0,$$

$$y > 0,$$

当 $t < 0$ 时,

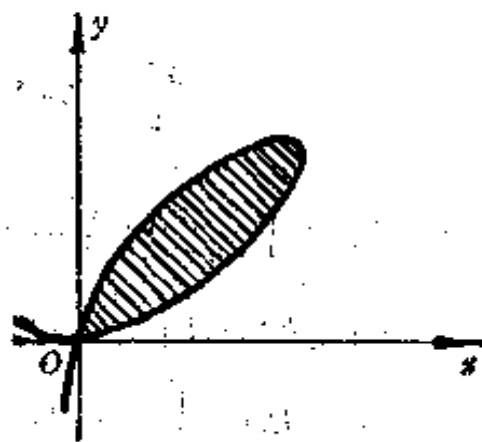
$$x < 0,$$

$$y > 0,$$

当 $t > 2$ 时,

$$x < 0,$$

$$y < 0.$$



如图4.27所示，所求的面积

图 4.27

$$\begin{aligned}
 S &= - \int_0^2 (2t^2 - t^3) \cdot 2(1-t) dt \\
 &= - 2 \int_0^2 (t^4 - 3t^3 + 2t^2) dt
 \end{aligned}$$

$$= \frac{8}{15}$$

2415. $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ ($0 \leq t \leq 2\pi$) (圆的渐伸线) 及 $x = a$, $y \leq 0$.

解 所求面积为

$$S = - \int_0^{2\pi} a(\sin t - t \cos t) \cdot a t \cos t dt$$

$$= - \int_{AB} y dx$$

$$= a^2 \left(\frac{1}{6} t^3 + \frac{1}{4} t^2 \sin 2t + \frac{1}{2} t \cos 2t \right. \\ \left. - \frac{1}{4} \sin 2t \right) \Big|_0^{2\pi} - \int_{AB} y dx$$

$$= \frac{a^2}{3} (4\pi^3 + 3\pi) - \int_{AB} y dx$$

其中 $\int_{AB} y dx$ 表沿着从点 $A(a, -2\pi a)$ 到点 $B(a, 0)$ 的直线 \overline{AB} 上的积分. 由于在 \overline{AB} 上 $x \equiv a$, 故 $dx = 0$, 从而 $\int_{AB} y dx = 0$. 于是, 得

$$S = \frac{a^2}{3} (4\pi^3 + 3\pi).$$

2416. $x = a(2 \cos t - \cos 2t)$, $y = a(2 \sin t - \sin 2t)$.

解 所求面积为

$$S = \frac{1}{2} \int_0^{2\pi} (x y'_t - y x'_t) dt$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^{2\pi} [a(2 \cos t - \cos 2t) \cdot a(2 \cos t \\
&\quad - 2 \cos 2t) - a(2 \sin t - \sin 2t) \\
&\quad \cdot a(-2 \sin t + 2 \sin 2t)] dt \\
&= 3a^2 \int_0^{2\pi} (1 - \cos t \cos 2t - \sin t \sin 2t) dt \\
&= 3a^2 \int_0^{2\pi} (1 - \cos t) dt = 6\pi a^2
\end{aligned}$$

2417. $x = \frac{c^2}{a} \cos^3 t$, $y = \frac{c^2}{b} \sin^3 t$ ($c^2 = a^2 - b^2$) (椭圆
的渐屈线)。

解 如图4.28所示。所求的面积为

$$\begin{aligned}
S &= 4 \int_0^{\frac{\pi}{2}} \frac{c^2}{b} \sin^3 t \cdot \frac{3c^2}{a} \cos^2 t \sin t dt \\
&= \frac{12c^4}{ab} \int_0^{\frac{\pi}{2}} \sin^4 t (1 - \sin^2 t) dt \\
&= \frac{3\pi c^4}{8ab}
\end{aligned}$$

求由下列极坐标
方程式所表曲线
围成的面积 S ;

2418. $r^2 = a^2 \cos 2\varphi$

(双纽线)。

解 如图4.29所
示, 所求的面积
为

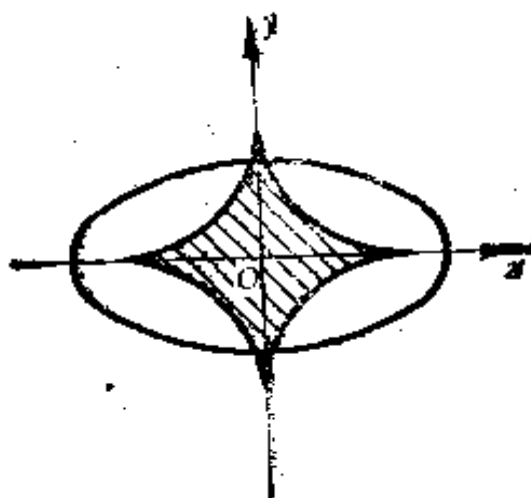


图 4.28



图 4.29

$$S = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \cos 2\varphi \, d\varphi = a^2.$$

2419. $r = a(1 + \cos \varphi)$ (心脏形线).

解 如图4.30所示, 所求的面积为

$$S = 2 \cdot \frac{1}{2} \int_0^{\pi} a^2 (1 + \cos \varphi)^2 \, d\varphi = \frac{3}{2} \pi a^2.$$

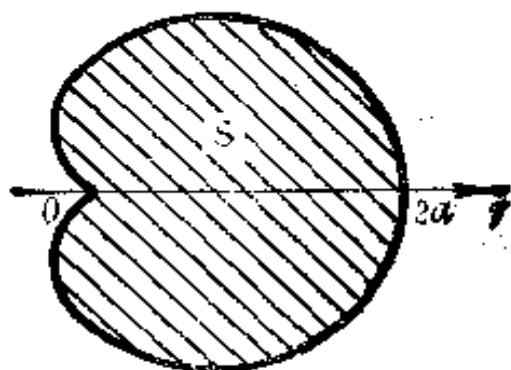


图 4.30

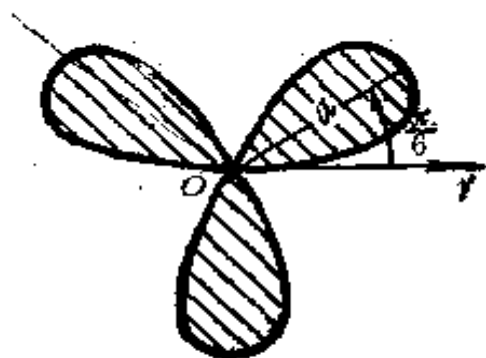


图 4.31

2420. $r = a \sin 3\varphi$ (三叶线)

解 如图4.31所示, 所求的面积为

$$S = 6 \cdot \frac{1}{2} \int_0^{\frac{\pi}{6}} a^2 \sin^2 3\varphi \, d\varphi = \frac{\pi a^2}{4}.$$

2421. $r = \frac{p}{1 - \cos \varphi}$ (抛物线), $\varphi = \frac{\pi}{4}$, $\varphi = \frac{\pi}{2}$.

解 所求的面积为

$$\begin{aligned}
S &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{p^2}{(1 - \cos \varphi)^2} d\varphi \\
&= \frac{p^2}{4} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{csc}^4 \frac{\varphi}{2} d\left(\frac{\varphi}{2}\right) \\
&= -\frac{p^2}{4} \left(\operatorname{ctg} \frac{\varphi}{2} + \frac{1}{3} \operatorname{ctg}^3 \frac{\varphi}{2} \right) \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&= \frac{p^2}{6} (4\sqrt{2} + 3)^*.
\end{aligned}$$

$$*) \quad \operatorname{ctg} \frac{\pi}{8} = 1 + \sqrt{2}.$$

2422. $r = \frac{p}{1 + \varepsilon \cos \varphi}$ ($0 < \varepsilon < 1$) (椭圆).

解 所求的面积为

$$\begin{aligned}
S &= 2 \cdot \frac{1}{2} \int_0^{\pi} \frac{p^2 d\varphi}{(1 + \varepsilon \cos \varphi)^2} \\
&= p^2 \int_0^{\pi} \frac{d\varphi}{(1 + \varepsilon \cos \varphi)^2}.
\end{aligned}$$

设

$$\operatorname{tg} \frac{\varphi}{2} = t,$$

并记

$$a^2 = \frac{1 + \varepsilon}{1 - \varepsilon},$$

则有

$$\int \frac{d\varphi}{(1 + \varepsilon \cos \varphi)^2} = \int \frac{2(t^2 + 1)dt}{(1 - \varepsilon)^2 (t^2 + a^2)^2}$$

$$\begin{aligned}
&= \frac{2}{(1-e)^2} \int \frac{dt}{t^2+a^2} \\
&\quad + \frac{2(1-a^2)}{(1-e)^2} \int \frac{dt}{(t^2+a^2)^2} \\
&= \frac{2}{a(1-e)^2} \operatorname{arc\,tg} \frac{t}{a} \\
&\quad + \frac{2(1-a^2)}{(1-e)^2} \left\{ \frac{t}{2a^2(t^2+a^2)} \right. \\
&\quad \left. + \frac{1}{2a^3} \operatorname{arc\,tg} \frac{t}{a} \right\}^* + C.
\end{aligned}$$

当 $0 \leq \varphi \leq \pi$ 时, $0 \leq t < +\infty$, 从而得一广义积分. 于是, 经计算得

$$\begin{aligned}
S &= \left\{ \frac{\pi}{a(1-e)^2} + \frac{(1-a^2)\pi}{2a^3(1-e)^2} \right\} \cdot p^2 \\
&= \frac{\pi p^2}{(1-e^2)^2}.
\end{aligned}$$

*) 利用 1921 题的递推公式.

2423. $r = a \cos \varphi$, $r = a(\cos \varphi + \sin \varphi)$ ($M(\frac{a}{2}, 0) \in S$).

解 如图 4.32 所示,

$$|OA| = a,$$

$$\alpha = -\frac{\pi}{4},$$

阴影部分即为所求的面积.

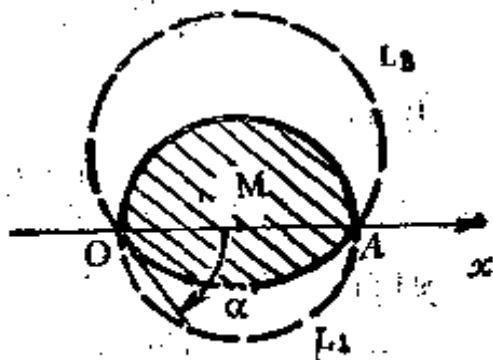


图 4.32

曲线 $L_1: r = a \cos \varphi,$

$L_2: r = a(\cos \varphi + \sin \varphi).$

所求的面积为

$$\begin{aligned} S &= \frac{\pi}{2} \left(\frac{a}{2} \right)^2 + \frac{1}{2} \int_{-\frac{\pi}{4}}^0 a^2 (\cos \varphi + \sin \varphi)^2 d\varphi \\ &= \frac{a^2(\pi - 1)}{4}. \end{aligned}$$

2424⁺. 求由曲线 $\varphi = r \operatorname{arc} \operatorname{tg} r$ 及二射线 $\varphi = 0$ 及 $\varphi = \frac{\pi}{\sqrt{3}}$

所围成之扇形的面积.

解 当 φ 由 0 变到 $\frac{\pi}{\sqrt{3}}$, r 从 0 变到 $\sqrt{3}$, 而

$$d\varphi = \left(\frac{r}{1+r^2} + \operatorname{arc} \operatorname{tg} r \right) dr.$$

所求的面积为

$$\begin{aligned} S &= \frac{1}{2} \int_0^{\frac{\pi}{\sqrt{3}}} r^2 d\varphi \\ &= \frac{1}{2} \int_0^{\sqrt{3}} \left(\frac{r^3}{1+r^2} + r^2 \operatorname{arc} \operatorname{tg} r \right) dr \\ &= \left[\frac{1}{6} r^2 - \frac{1}{6} \ln(1+r^2) + \frac{1}{6} r^3 \operatorname{arc} \operatorname{tg} r \right] \Big|_0^{\sqrt{3}} \\ &= \frac{1}{2} - \frac{1}{3} \ln 2 + \frac{\sqrt{3}}{6} \pi. \end{aligned}$$

2425. 求封闭曲线

$$r = \frac{2at}{1+t^2}, \quad \varphi = \frac{\pi t}{1+t}$$

所包围的面积。

解 当曲线封闭时, t 由 0 变化到 $+\infty$ 。所求的面积
为

$$\begin{aligned}
 S &= \frac{1}{2} \int_0^{+\infty} r^2 d\varphi = 2\pi a^2 \int_0^{+\infty} \frac{t^2}{(1+t^2)^2(1+t)^2} dt \\
 &= 2\pi a^2 \lim_{t \rightarrow +\infty} \left\{ \int_0^t \frac{dt}{4(1+t)^2} - \frac{1}{4} \int_0^t \frac{dt}{1+t^2} \right. \\
 &\quad \left. + \frac{1}{2} \int_0^t \frac{t dt}{(1+t^2)^2} \right\} \\
 &= 2\pi a^2 \lim_{t \rightarrow +\infty} \left\{ -\frac{1}{4(1+t)} - \frac{1}{4} \operatorname{arc} \operatorname{tg} t \right. \\
 &\quad \left. - \frac{1}{4} \cdot \frac{1}{1+t^2} \right\} \Big|_0^t \\
 &= \pi a^2 \left(1 - \frac{\pi}{4} \right).
 \end{aligned}$$

变为极坐标, 以求下列曲线所围成的面积:

2426. $x^3 + y^3 = 3axy$ (笛卡尔叶形线)。

解 $r^3(\cos^3\varphi + \sin^3\varphi) = 3ar^2\cos\varphi\sin\varphi$,

于是

$$r = \frac{3a \sin\varphi \cos\varphi}{\sin^3\varphi + \cos^3\varphi}.$$

当 $\varphi \in \left[0, \frac{\pi}{2} \right]$ 时, $r \geq 0$, 且当 $\varphi = 0$ 及 $\varphi = \frac{\pi}{2}$

时, $r = 0$ 。所以, 从 $\varphi = 0$ 到 $\varphi = \frac{\pi}{2}$, 叶形线位于第一象限部分所围成的面积, 即为所要求的面积 (图

4.33)

$$S = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{9a^2 \sin^2 \varphi \cos^2 \varphi}{(\sin^2 \varphi + \cos^2 \varphi)^2} d\varphi$$

$$= \frac{9a^2}{2} \int_0^{+\infty} \frac{t^2 dt}{(1+t^2)^2} \quad *)$$

$$= \frac{9a^2}{2} \lim_{b \rightarrow +\infty} \left. \frac{-1}{3(1+t^2)} \right|_0^b$$

$$= \frac{3a^2}{2}$$

*) 设 $\operatorname{tg} \varphi = t$.

2427. $x^4 + y^4 = a^2(x^2 + y^2)$.

解 $r^4(\sin^4 \varphi + \cos^4 \varphi) = a^2 r^2$,

于是

$$r = \frac{\sqrt{2} a}{\sqrt{2 - \sin^2 2\varphi}}$$

如图4.34所示, 所求的面积为

$$S = 8 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{2a^2}{2 - \sin^2 2\varphi} d\varphi$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} \frac{1}{2 - \sin^2 t} dt$$

$$= \frac{2a^2}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2} - \sin t} + \frac{1}{\sqrt{2} + \sin t} \right) dt$$

$$= \sqrt{2} a^2 \left\{ 2 \operatorname{arc} \operatorname{tg} \left(\sqrt{2} \operatorname{tg} \frac{t}{2} - 1 \right) \right\}$$

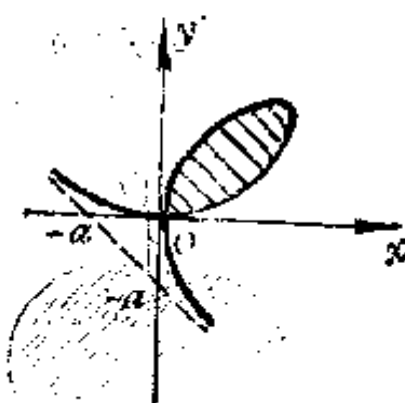


图 4.33

$$\begin{aligned}
 & + 2 \operatorname{arc} \operatorname{tg} \left(\sqrt{2} \operatorname{tg} \frac{t}{2} + 1 \right) \Big|_0^{\frac{\pi}{2}} \\
 & = 2\sqrt{2}a^2 \{ \operatorname{arc} \operatorname{tg}(\sqrt{2}-1) + \operatorname{arc} \operatorname{tg}(\sqrt{2}+1) \} \\
 & = 2\sqrt{2}a^2 \cdot \frac{\pi}{2} = \sqrt{2}\pi a^2.
 \end{aligned}$$

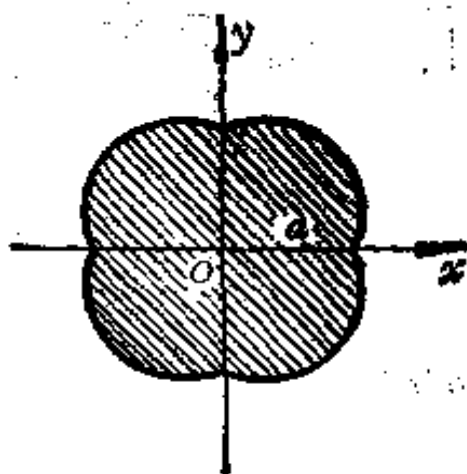


图 4.34

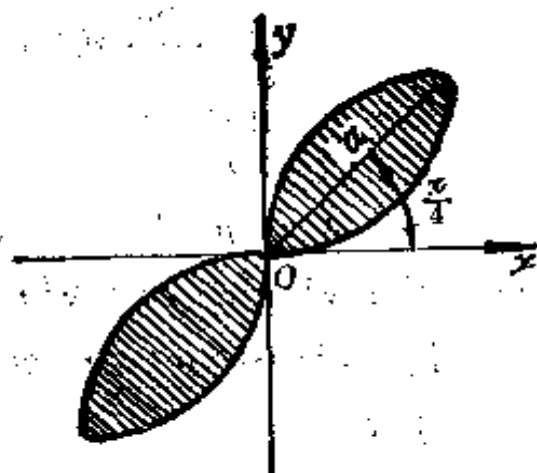


图 4.35

2428. $(x^2 + y^2)^2 = 2a^2xy$ (双纽线).

解 $r^2 = a^2 \sin 2\varphi$ (图4.35).

所求的面积为

$$S = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \sin 2\varphi = a^2.$$

化方程式为参数式的形状, 以求下列曲线所围成的面积:

2429. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (内摆线).

解 设

$$x = a \cos^3 t, \quad y = a \sin^3 t,$$

其中 $0 \leq t \leq \frac{\pi}{2}$, 它对应于四分之一的面积。所求的面积为其四倍, 即

$$\begin{aligned} S &= 4 \int_0^{\frac{\pi}{2}} y dx = 4 \int_0^{\frac{\pi}{2}} (-3a^2 \sin^4 t \cos^2 t) dt \\ &= 12a^2 \int_0^{\frac{\pi}{2}} (\sin^4 t - \sin^6 t) dt = \frac{3\pi a^2}{8}. \end{aligned}$$

2430. $x^4 + y^4 = ax^2 y$.

解 设

$$y = tx,$$

则曲线的参数方程为

$$\begin{cases} x = \frac{at}{1+t^4}, \\ y = \frac{at^2}{1+t^4}. \end{cases} \quad (-\infty < t < +\infty)$$

利用对称性知, 所求的面积为

$$\begin{aligned} S &= -2 \int_0^{+\infty} \frac{at^2}{1+t^4} \cdot \frac{a(1-3t^4)}{(1+t^4)^2} dt \\ &= -2a^2 \left(\int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt \right. \\ &\quad \left. - 3 \int_0^{+\infty} \frac{t^6}{(1+t^4)^3} dt \right). \end{aligned}$$

因为

$$\int \frac{x^n dx}{(a+bx^4)^m}$$

$$= \frac{x^{n-3}}{(n+1-4m)b \cdot (a+bx^4)^{n-1}} - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4}}{(a+bx^4)^n} dx^{**},$$

所以

$$\begin{aligned} & \int_0^{+\infty} \frac{t^6}{(1+t^4)^3} dt \\ &= -\frac{t^3}{5(1+t^4)^2} \Big|_0^{+\infty} + \frac{3}{5} \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt \\ &= \frac{3}{5} \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt, \end{aligned}$$

于是

$$S = \frac{8}{5} a^2 \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt.$$

又因

$$\begin{aligned} & \int \frac{x^n dx}{(a+bx^4)^m} \\ &= \frac{x^{n+1}}{4a(m-1)(a+bx^4)^{m-1}} \\ &+ \frac{4m-n-5}{4a(m-1)} \int \frac{x^n dx}{(a+bx^4)^{m-1}}, \quad (***) \end{aligned}$$

所以

$$\begin{aligned} & \int_0^{+\infty} \frac{t^2}{(1+t^4)^3} dt \\ &= \frac{t^3}{8(1+t^4)^2} \Big|_0^{+\infty} + \frac{5}{8} \int_0^{+\infty} \frac{t^2}{(1+t^4)^2} dt \end{aligned}$$

$$= \frac{5}{8} \int_0^{+\infty} \frac{t^2}{(1+t^4)^2} dt$$

$$= \frac{5}{8} \left[\frac{t^3}{4(1+t^4)} \Big|_0^{+\infty} + \frac{1}{4} \int_0^{+\infty} \frac{t^2 dt}{1+t^4} \right]$$

$$= \frac{5}{32} \int_0^{+\infty} \frac{t^2}{1+t^4} dt,$$

于是

$$S = \frac{1}{4} a^2 \int_0^{+\infty} \frac{t^2}{1+t^4} dt.$$

利用

$$\int \frac{x^2}{a+bx^4} dx$$

$$= \frac{1}{4b \cdot \sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}} \left\{ \ln \frac{x^2 - \sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}x + \sqrt{\frac{a}{b}}}{x^2 + \sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}x + \sqrt{\frac{a}{b}}} \right.$$

$$\left. + 2 \operatorname{arc} \operatorname{tg} \frac{\sqrt[4]{\frac{a}{b}} \cdot \sqrt{2}x}{\sqrt{\frac{a}{b}} - x^2} \right\} \quad (ab > 0),$$

即得

$$\int \frac{t^2}{1+t^4} dt$$

$$= \frac{1}{4\sqrt{2}} \left\{ \ln \frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} \right.$$

$$\left. + 2 \operatorname{arc} \operatorname{tg} \frac{\sqrt{2}t}{1-t^2} \right\} + C.$$

考虑到上述式子右端的函数 $\arctg \frac{\sqrt{2}t}{1-t^2}$ 在 $(0, +\infty)$ 中的 $t=1$ 点间断, 并且

$$\lim_{t \rightarrow 1+0} \arctg \frac{\sqrt{2}t}{1-t^2} = -\frac{\pi}{2},$$

及

$$\lim_{t \rightarrow 1-0} \arctg \frac{\sqrt{2}t}{1-t^2} = \frac{\pi}{2}.$$

于是

$$\begin{aligned} \int_0^{+\infty} \frac{t^2}{1+t^4} dt &= \int_0^1 \frac{t^2}{1+t^4} dt + \int_1^{+\infty} \frac{t^2}{1+t^4} dt \\ &= \frac{1}{4\sqrt{2}} \left\{ \ln \frac{t^2 + \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} + 2 \arctg \frac{\sqrt{2}t}{1-t^2} \right\} \Big|_0^1 \\ &\quad + \frac{1}{4\sqrt{2}} \left\{ \ln \frac{t^2 + \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} \right. \\ &\quad \left. + 2 \arctg \frac{\sqrt{2}t}{1-t^2} \right\} \Big|_1^{+\infty} \\ &= -\frac{2}{4\sqrt{2}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\sqrt{2}\pi}{4}, \end{aligned}$$

最后得所求的面积为

$$S = \frac{\sqrt{2}\pi}{16} a^2.$$

*) 参阅“函数表与积分表”(И.М.雷日克, И.С.格拉德什坦)第64页“(2.133)2”.

***) 参阅同书第64页“(2.133)1”。

****) 参阅同书第64页“(2.132)3”。

§6. 弧长的计算法

1° 在直角坐标系中的弧长 平滑(连续可微分的)曲线

$$y = y(x) \quad (a \leq x \leq b)$$

上一段弧的长度等于

$$s = \int_a^b \sqrt{1 + y'^2(x)} dx.$$

2° 参数方程所表曲线的弧长 若曲线C用参数方程式给出

$$x = x(t), \quad y = y(t) \quad (t_0 \leq t \leq T),$$

式中 $x(t)$, $y(t)$ 为在闭区间 $[t_0, T]$ 内可微分的连续函数, 则曲线C的弧长等于

$$s = \int_{t_0}^T \sqrt{x'^2(t) + y'^2(t)} dt.$$

3° 极坐标系中的弧长 若

$$r = r(\varphi) \quad (\alpha \leq \varphi \leq \beta),$$

式中 $r(\varphi)$ 及其导函数 $r'(\varphi)$ 在闭区间 $[\alpha, \beta]$ 上皆是连续的, 则曲线上对应的一段弧长等于

$$s = \int_{\alpha}^{\beta} \sqrt{r^2(\varphi) + r'^2(\varphi)} d\varphi.$$

关于空间曲线的弧长可参阅第八章。

求下列曲线的弧长。

2431. $y = x^{\frac{3}{2}} (0 \leq x \leq 4)$.

解 所求的弧长为

$$s = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx = \frac{8}{27}(10\sqrt{10} - 1).$$

2432. $y^2 = 2px (0 \leq x \leq x_0)$.

$$\begin{aligned} \text{解 } y' &= \frac{p}{y}, \quad \sqrt{1 + y'^2} = \sqrt{1 + \frac{p^2}{y^2}} = \sqrt{1 + \frac{p}{2x}} \\ &= \frac{1}{\sqrt{2}} \frac{\sqrt{p+2x}}{\sqrt{x}}. \end{aligned}$$

所求的弧长为

$$\begin{aligned} s &= 2 \int_0^{x_0} \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{p+2x}}{\sqrt{x}} dx \\ &= 2\sqrt{2} \int_0^{x_0} \sqrt{p+2x} d(\sqrt{x}) \\ &= 2\sqrt{2} \left\{ \frac{1}{2} \sqrt{x(p+2x)} \right. \\ &\quad \left. + \frac{p}{2\sqrt{2}} \ln \left(\sqrt{x} + \sqrt{x + \frac{p}{2}} \right) \right\} \Big|_0^{x_0} \\ &= 2\sqrt{x_0 \left(x_0 + \frac{p}{2} \right)} \\ &\quad + p \ln \left(\frac{\sqrt{x_0} + \sqrt{x_0 + \frac{p}{2}}}{\sqrt{\frac{p}{2}}} \right). \end{aligned}$$

2433. $y = a \operatorname{ch} \frac{x}{a}$ 从点 $A(0, a)$ 至点 $B(b, h)$.

解 所求的弧长为

$$\begin{aligned} s &= \int_0^b \sqrt{1 + \operatorname{sh}^2 \frac{x}{a}} dx = \int_0^b \operatorname{ch} \frac{x}{a} dx \\ &= a \operatorname{sh} \frac{x}{a} \Big|_0^b = a \operatorname{sh} \frac{b}{a} = \sqrt{h^2 - a^2} \end{aligned}$$

$$\begin{aligned} *) \text{ 由于 } h &= a \operatorname{ch} \frac{b}{a}, \text{ 故 } \operatorname{sh} \frac{b}{a} = \sqrt{\operatorname{ch}^2 \frac{b}{a} - 1} \\ &= \frac{1}{a} \sqrt{h^2 - a^2}. \end{aligned}$$

2434. $y = e^x$ ($0 \leq x \leq x_0$).

解 所求的弧长为

$$\begin{aligned} s &= \int_0^{x_0} \sqrt{1 + e^{2x}} dx \\ &= \left(\sqrt{1 + e^{2x}} + \frac{1}{2} \ln \frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1} \right) \Big|_0^{x_0} \\ &= \sqrt{1 + e^{2x_0}} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{1 + e^{2x_0}} - 1}{\sqrt{1 + e^{2x_0}} + 1} \\ &\quad - \frac{1}{2} \ln \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \\ &= x_0 - \sqrt{2} + \sqrt{1 + e^{2x_0}} \\ &\quad - \ln \frac{1 + \sqrt{1 + e^{2x_0}}}{1 + \sqrt{2}}. \end{aligned}$$

$$2435. \quad x = \frac{1}{4} y^2 - \frac{1}{2} \ln y \quad (1 \leq y \leq e).$$

解 所求的弧长为

$$\begin{aligned} s &= \int_1^e \sqrt{1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2} dy \\ &= \int_1^e \frac{1+y^2}{2y} dy = \frac{e^2+1}{4}. \end{aligned}$$

$$2436. \quad y = a \ln \frac{a^2}{a^2-x^2} \quad (0 \leq x \leq b < a).$$

$$\text{解 } y' = \frac{2ax}{a^2-x^2}, \quad \sqrt{1+y'^2} = \frac{a^2+x^2}{a^2-x^2}.$$

所求的弧长为

$$s = \int_0^b \frac{a^2+x^2}{a^2-x^2} dx = a \ln \frac{a+b}{a-b} - b.$$

$$2437. \quad y = \ln \cos x \quad (0 \leq x \leq a < \frac{\pi}{2}).$$

解 所求的弧长为

$$\begin{aligned} s &= \int_0^a \sqrt{1 + \operatorname{tg}^2 x} dx \\ &= \int_0^a \frac{dx}{\cos x} = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right). \end{aligned}$$

$$2438. \quad x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \quad (0 < b \leq y \leq a).$$

$$\text{解 } \frac{dx}{dy} = -\frac{\sqrt{a^2 - y^2}}{y}, \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{a}{y}.$$

所求的弧长为

$$s = \int_b^a \frac{a}{y} dy = a \ln \frac{a}{b}.$$

2439. $y^2 = \frac{x^3}{2a-x} \quad \left(0 \leq x \leq \frac{5}{3}a \right)^{**})$.

解 如图4.36所示.

设 $y = tx$, 得

$$\begin{cases} x = \frac{2at^2}{1+t^2}, \\ y = \frac{2at^3}{1+t^2}. \end{cases}$$

当 $0 \leq x \leq \frac{5}{3}a$ 时, $0 \leq t$

$\leq \sqrt{5}$ (一半弧长).

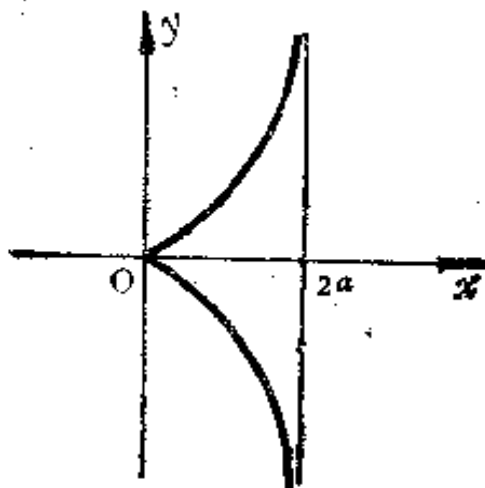


图 4.36

$$x_t' = \frac{4at}{(t^2+1)^2}, \quad y_t' = \frac{2at^4 + 6at^2}{(t^2+1)^2},$$

$$\sqrt{x_t'^2 + y_t'^2} = \frac{2at\sqrt{t^2+4}}{t^2+1}.$$

所求的弧长为

$$s = \int_0^{\sqrt{5}} 2 \frac{2at\sqrt{t^2+4}}{t^2+1} dt$$

$$= 32a \int_0^{\arcsin \frac{\sqrt{5}}{2}} \frac{\sin \theta d\theta}{\cos^2 \theta (1+3\sin^2 \theta)} \quad (**)$$

$$= \frac{32a}{3} \int_1^{\frac{3}{2}} \frac{dz}{z^2(z^2 - \frac{4}{3})} \quad (***)$$

$$= \frac{32a}{3} \left\{ \frac{3}{4} \cdot \frac{1}{z} + \frac{3\sqrt{3}}{16} \ln \frac{z - \frac{2}{\sqrt{3}}}{z + \frac{2}{\sqrt{3}}} \right\} \Big|_1^{\frac{1}{2}}$$

$$= 4a \left(1 + 3\sqrt{3} \ln \frac{1 + \sqrt{3}}{\sqrt{2}} \right).$$

*) 原题误为 $y^2 = \frac{x^2}{2a-x}$, 现按原答案予以改正.

**) 设 $t = 2 \operatorname{tg} \theta$.

***) 设 $z = \cos \theta$.

2440. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (内摆线).

$$\text{解 } y' = -\sqrt[3]{\frac{y}{x}}, \sqrt{1+y'^2} = \left(\frac{a}{x}\right)^{\frac{1}{3}}.$$

所求的弧长为

$$s = 4 \int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}} dx = 6a.$$

2441. $x = \frac{c^2}{a} \cos^3 t, y = \frac{c^2}{b} \sin^3 t, c^2 = a^2 - b^2$ (椭圆的渐屈线).

解 $\sqrt{x_1'^2 + y_1'^2}$

$$= \frac{3c^2}{ab} \sin t \cos t \sqrt{b^2 \cos^2 t + a^2 \sin^2 t}.$$

所求的弧长为

$$s = 4 \int_0^{\frac{\pi}{2}} \frac{3c^2}{ab} \sin t \cos t \sqrt{b^2 \cos^2 t + a^2 \sin^2 t} dt$$

$$\begin{aligned}
&= \frac{12c^2}{3ab(a^2 - b^2)} \left\{ b^2 + (a^2 - b^2) \sin^2 t \right\}^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} \\
&= \frac{4(a^3 - b^3)}{ab}.
\end{aligned}$$

2442⁺. $x = a \cos^4 t$, $y = a \sin^4 t$.

解 $\sqrt{x_t'^2 + y_t'^2} = 4a \sin t \cos t \sqrt{\cos^4 t + \sin^4 t}$.

所求的弧长为

$$\begin{aligned}
s &= \int_0^{\frac{\pi}{2}} 4a \sin t \cos t \sqrt{\cos^4 t + \sin^4 t} dt \\
&= 2a \int_0^{\frac{\pi}{2}} \sqrt{2 \left(\sin^2 t - \frac{1}{2} \right)^2 + \frac{1}{2}} d \left(\sin^2 t - \frac{1}{2} \right) \\
&= 2a \left[\frac{\sin^2 t - \frac{1}{2}}{2} \sqrt{\cos^4 t + \sin^4 t} \right. \\
&\quad \left. + \frac{1}{4\sqrt{2}} \ln \left| \sin^2 t - \frac{1}{2} + \sqrt{\frac{1}{2}(\cos^4 t + \sin^4 t)} \right| \right] \Big|_0^{\frac{\pi}{2}} \\
&= \left[1 + \frac{1}{\sqrt{2}} \ln(1 + \sqrt{2}) \right] a.
\end{aligned}$$

2443. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$).

解 所求的弧长为

$$s = \int_0^{2\pi} \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt$$

$$= 2a \int_0^{2\pi} \sin \frac{t}{2} dt = 8a.$$

2444. $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ ($0 \leq t \leq 2\pi$) (圆的渐伸线).

解 $x'_t = a t \cos t$, $y'_t = a t \sin t$,

$$\sqrt{x'_t{}^2 + y'_t{}^2} = at.$$

所求的弧长为

$$s = \int_0^{2\pi} at dt = 2\pi^2 a.$$

2445⁺. $x = a(\operatorname{sh} t - t)$, $y = a(\operatorname{ch} t - 1)$ ($0 \leq t \leq T$).

解 $\sqrt{x'_t{}^2 + y'_t{}^2} = \sqrt{2} a \cdot \sqrt{\operatorname{ch}^2 t - \operatorname{ch} t}$.

所求的弧长为

$$s = \int_0^T \sqrt{2} a \sqrt{\operatorname{ch}^2 t - \operatorname{ch} t} dt$$

$$= \sqrt{2} a \int_1^{\operatorname{ch} T} \sqrt{\frac{\theta}{\theta+1}} d\theta^{**}$$

$$= 2\sqrt{2} a \int_{\frac{\pi}{4}}^{\operatorname{arctg} \sqrt{\operatorname{ch} T}} \frac{\sin^2 z}{\cos^3 z} dz^{**}$$

$$= 2\sqrt{2} a \left\{ \frac{\sin z}{2 \cos^2 z} - \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{z}{2} \right) \right\} \Big|_{\frac{\pi}{4}}^{\operatorname{arctg} \sqrt{\operatorname{ch} T}}$$

$$= \sqrt{2} a (\sqrt{\operatorname{ch} T} \cdot \sqrt{1 + \operatorname{ch} T} - \sqrt{2})$$

$$- \sqrt{2} a (\ln(\sqrt{\operatorname{ch} T} + \sqrt{1 + \operatorname{ch} T})$$

$$- \ln(1 + \sqrt{2}))]$$

$$= 2a \left(\operatorname{ch} \frac{T}{2} \cdot \sqrt{\operatorname{ch} T + 1} \right)$$

$$-\sqrt{2} a \ln \frac{\sqrt{2} \operatorname{ch} \frac{T}{2} + \sqrt{\operatorname{ch} T}}{\sqrt{2} + 1} \quad (***)$$

*) 设 $\theta = \operatorname{ch} t$,

***) 设 $\theta = \operatorname{tg}^2 z$.

***) $\sqrt{1 + \operatorname{ch} T} = \sqrt{2} \operatorname{ch} \frac{T}{2}$.

2446. $r = a\varphi$ (阿基米得螺线) ($0 \leq \varphi \leq 2\pi$).

解 所求的弧长为

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{a^2 \varphi^2 + a^2} d\varphi \\ &= a \left\{ \frac{\varphi}{2} \sqrt{\varphi^2 + 1} + \frac{1}{2} \ln(\varphi + \sqrt{\varphi^2 + 1}) \right\} \Big|_0^{2\pi} \\ &= a \left\{ \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln(2\pi + \sqrt{1 + 4\pi^2}) \right\}. \end{aligned}$$

2447. $r = ae^{m\varphi}$ ($m > 0$) 当 $0 < r < a$.

解 $0 < r < a$, $-\infty < \varphi < 0$.

所求的弧长为

$$\begin{aligned} s &= \int_{-\infty}^0 \sqrt{a^2 e^{2m\varphi} + a^2 m^2 e^{2m\varphi}} d\varphi \\ &= a \sqrt{m^2 + 1} \int_{-\infty}^0 e^{m\varphi} d\varphi \\ &= \lim_{a \rightarrow -\infty} \int_a^0 e^{m\varphi} d\varphi = \frac{a \sqrt{1 + m^2}}{m}. \end{aligned}$$

2448. $r = a(1 + \cos \varphi)$.

解 $\sqrt{r^2 + r'^2} = 2a \cos \frac{\varphi}{2}$.

所求的弧长为

$$s = 2 \int_0^{\pi} 2a \cos \frac{\varphi}{2} d\varphi = 8a.$$

2449. $r = \frac{p}{1 + \cos \varphi} \left(|\varphi| \leq \frac{\pi}{2} \right)$.

解 $r' = \frac{p \sin \varphi}{(1 + \cos \varphi)^2}$,

$$\sqrt{r^2 + r'^2} = \frac{2p \cos \frac{\varphi}{2}}{(1 + \cos \varphi)^2}.$$

所求的弧长为

$$\begin{aligned} s &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2p \cos \frac{\varphi}{2}}{(1 + \cos \varphi)^2} d\varphi = \frac{p}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^3 \frac{\varphi}{2} d\varphi \\ &= \frac{p}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec \frac{\varphi}{2} (1 + \operatorname{tg}^2 \frac{\varphi}{2}) d\varphi \\ &= p \left\{ \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos \frac{\varphi}{2}} \right. \\ &\quad \left. + 2 \int_0^{\frac{\pi}{2}} \sqrt{\sec^2 \frac{\varphi}{2} - 1} d \left(\sec \frac{\varphi}{2} \right) \right\} \\ &= 2p \left\{ \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{\varphi}{4} \right) + \frac{\sec \frac{\varphi}{2}}{2} \sqrt{\sec^2 \frac{\varphi}{2} - 1} \right. \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \ln \left(\sec \frac{\varphi}{2} + \operatorname{tg} \frac{\varphi}{2} \right) \Big|_0^{\frac{\pi}{2}} \\
 & = p \{ \sqrt{2} + \ln(\sqrt{2} + 1) \}.
 \end{aligned}$$

2450. $r = a \sin^3 \frac{\varphi}{3}$.

解 $\sqrt{r^2 + r'^2} = a \sin^2 \frac{\varphi}{3} (0 \leq \varphi \leq 3\pi)$ (图 4.37).

所求的弧长为

$$\begin{aligned}
 s &= \int_0^{3\pi} a \sin^2 \frac{\varphi}{3} d\varphi \\
 &= \frac{3\pi a}{2}.
 \end{aligned}$$

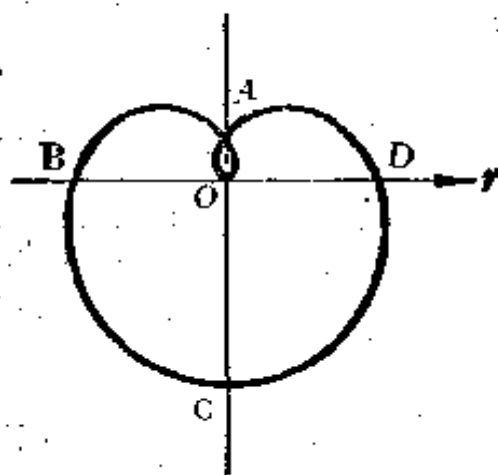


图 4.37

我们甚至可以证明

1° 弧 \widehat{AB} 为弧 \widehat{OABC} 的三分之一;

2° \widehat{OA} , \widehat{AB} , \widehat{BC} 之间

依次是等差的, 其公差为 $\frac{3a}{8} \sqrt{3}$.

不仅如此, 我们还可证明更一般的情况:

曲线 $r = a \sin^n \left(\frac{\theta}{n} \right)$ (n 为正整数) 之全长为

$$s = \begin{cases} \frac{(2k-2)!!}{(2k-1)!!} 4ka, & \text{当 } n=2k \text{ 时,} \\ \frac{(2k+1)!!}{(2k)!!} \pi a, & \text{当 } n=2k+1 \text{ 时.} \end{cases}$$

2451. $r = a \operatorname{th} \frac{\varphi}{2} \quad (0 \leq \varphi \leq 2\pi)$,

解 $r' = \frac{a}{2} \cdot \frac{1}{\operatorname{ch}^2 \frac{\varphi}{2}}$,

$$\sqrt{r^2 + r'^2} = \frac{a}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \sqrt{4 \operatorname{sh}^2 \frac{\varphi}{2} \operatorname{ch}^2 \frac{\varphi}{2} + 1}$$

$$= \frac{a}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \sqrt{\operatorname{sh}^2 \varphi + 1}$$

$$= \frac{a \operatorname{ch} \varphi}{2 \operatorname{ch}^2 \frac{\varphi}{2}} = \frac{a \operatorname{ch} \varphi}{1 + \operatorname{ch} \varphi}$$

$$= a \left(1 - \frac{1}{1 + \operatorname{ch} \varphi} \right)$$

$$= a \left(1 - \frac{1}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \right)$$

所求的弧长为

$$s = \int_0^{2\pi} a \left(1 - \frac{1}{2 \operatorname{ch}^2 \frac{\varphi}{2}} \right) d\varphi = a \left(\varphi - \operatorname{th} \frac{\varphi}{2} \right) \Big|_0^{2\pi}$$

$$= a (2\pi - \operatorname{th} \pi).$$

2452. $\varphi = \frac{1}{2} \left(r + \frac{1}{r} \right) \quad (1 \leq r \leq 3)$.

解 $r^2 - 2r\varphi + 1 = 0$, 两边对 φ 求导函数, 得

$$2rr' - 2\varphi r' - 2r = 0$$

即

$$r' = \frac{r}{r-\varphi},$$

从而

$$\sqrt{r^2 + r'^2} = \frac{r\varphi}{r-\varphi} = \frac{r^3+r}{r^2-1},$$

$$d\varphi = \frac{1}{2}\left(1 - \frac{1}{r^2}\right)dr.$$

所求的弧长为

$$\begin{aligned} s &= \frac{1}{2} \int_1^3 \frac{r^3+r}{r^2-1} \cdot \frac{r^2-1}{r^2} dr \\ &= \frac{1}{2} \int_1^3 \left(r + \frac{1}{r}\right) dr = 2 + \frac{1}{2} \ln 3. \end{aligned}$$

2453. 证明: 椭圆

$$x = a \cos t, \quad y = b \sin t$$

的弧长等于正弦曲线 $y = c \sin \frac{x}{b}$ 的一波之长, 其中

$$c = \sqrt{a^2 - b^2}.$$

证 对于椭圆, 其全长为

$$\begin{aligned} s_1 &= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \\ &= \int_0^{2\pi} \sqrt{a^2 - c^2 \cos^2 t} dt \\ &= a \int_0^{2\pi} \sqrt{1 - e^2 \cos^2 t} dt \\ &= a \int_0^{2\pi} \sqrt{1 - e^2 \sin^2 t} dt. \end{aligned}$$

对于正弦曲线，其一波（ x 由0到 $2\pi b$ ）之长为

$$\begin{aligned} s_2 &= \int_0^{2\pi b} \sqrt{1 + \frac{c^2}{b^2} \cos^2 \frac{x}{b}} dx \\ &= \int_0^{2\pi} \sqrt{b^2 + c^2 \cos^2 t} dt \\ &= \int_0^{2\pi} \sqrt{a^2 - c^2 \sin^2 t} dt \\ &= a \int_0^{2\pi} \sqrt{1 - e^2 \sin^2 t} dt. \end{aligned}$$

所以 $s_1 = s_2$ ，本题得证。

2454. 抛物线 $4ay = x^2$ 沿 Ox 轴滚动。证明抛物线的焦点划成悬链线。

解 如图4.38所示，设抛物线切 Ox 轴于点 $A(s, 0)$ ， O' 为抛物线的顶点， P' 为焦点， $O'Y'$ 为对称轴， $O'X' \perp O'Y'$ 。过 A 作 $AB \perp O'X'$ 。

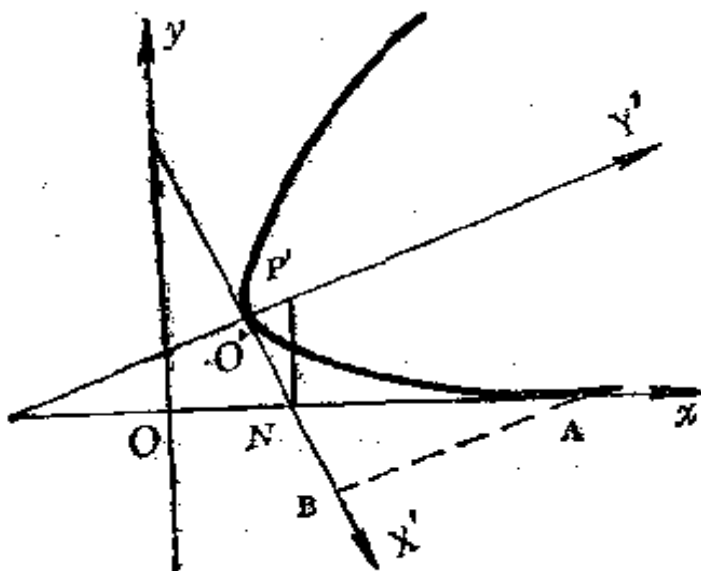


图 4.38

引入参数 $O'N = t$ ，则由抛物线的性质易知：
 $P'N \perp Ox$ ， $O'B = 2O'N = 2t$ 。从而有

$$AB = \frac{(2t)^2}{4a} = \frac{t^2}{a}, \quad AN = t \cdot \sqrt{1 + \frac{t^2}{a^2}}$$

$$s = \int_0^{2t} \sqrt{1 + \left(\frac{x}{2a}\right)^2} dx$$

$$= t \sqrt{1 + \left(\frac{t}{a}\right)^2} + a \ln \left(\frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2} \right),$$

$$P'N = a \sqrt{1 + \left(\frac{t}{a}\right)^2}.$$

于是，焦点 P' 的坐标 x ， y 由参数 t 表出：

$$\begin{cases} x = s - AN = a \ln \left(\frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2} \right), & (1) \end{cases}$$

$$\begin{cases} y = P'N = a \sqrt{1 + \left(\frac{t}{a}\right)^2}. & (2) \end{cases}$$

由 (1) 式得

$$e^{\frac{x}{a}} = \frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2},$$

$$e^{-\frac{x}{a}} = -\frac{t}{a} + \sqrt{1 + \left(\frac{t}{a}\right)^2}.$$

上面两式相加，得

$$e^{\frac{x}{a}} + e^{-\frac{x}{a}} = 2 \sqrt{1 + \left(\frac{t}{a}\right)^2}.$$

再以 (2) 式代入上式，最后得

$$y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) = a \operatorname{ch} \frac{x}{a}$$

这说明抛物线的焦点划成悬链线。

2455. 求环线

$$y = \pm \left(\frac{1}{3} - x \right) \sqrt{x}$$

所包围的面积与周长等于这曲线的围线长的圆面积之比。

解 当 $x=0$ 及 $x=\frac{1}{3}$ 时, $y=0$. 此环线的面积为

$$S_1 = 2 \int_0^{\frac{1}{3}} \left(\frac{1}{3} - x \right) \sqrt{x} dx = \frac{8}{135\sqrt{3}}$$

此环线的周长为

$$\begin{aligned} s &= 2 \int_0^{\frac{1}{3}} \sqrt{1 + \left(\frac{1}{6\sqrt{x}} - \frac{3\sqrt{x}}{2} \right)^2} dx \\ &= 2 \int_0^{\frac{1}{3}} \left(\frac{1}{6\sqrt{x}} - \frac{3\sqrt{x}}{2} \right) dx \\ &= \frac{4}{3\sqrt{3}} \end{aligned}$$

按题设有 $\frac{4}{3\sqrt{3}} = 2\pi R$, 所以 $R = \frac{2}{3\sqrt{3}\pi}$. 圆面

积

$$S_2 = \pi R^2 = \frac{4}{27\pi}$$

于是,

$$\frac{S_1}{S_2} = \frac{2\pi}{5\sqrt{3}} \approx 0.73.$$

§7. 体积的计算法

1° 由已知横切面计算物体体积 若物体的体积 V 存在及 $S = S(x)$ ($a \leq x \leq b$) 为用平面切下的物体的横断面积, 而此横断面为经过 x 点垂直于 Ox 轴者, 则

$$V = \int_a^b S(x) dx.$$

2° 旋转体的体积 面积

$$a \leq x \leq b; \quad 0 \leq y \leq y(x),$$

式中 $y(x)$ 为单值连续函数, 绕 Ox 轴旋转所成旋转体的体积等于

$$V_r = \pi \int_a^b y^2(x) dx.$$

更普遍的情形: 面积

$$a \leq x \leq b; \quad y_1(x) \leq y \leq y_2(x),$$

式中 $y_1(x)$ 和 $y_2(x)$ 是非负的连续函数, 绕 Ox 轴旋转所成的环形的体积等于

$$V = \pi \int_a^b [y_2^2(x) - y_1^2(x)] dx.$$

2456. 求顶楼的体积, 其底是边长等于 a 及 b 的矩形, 其顶的棱边等于 c , 而高等于 h .

解 如图4.39所示的顶楼, 取 x 轴向下, 则有

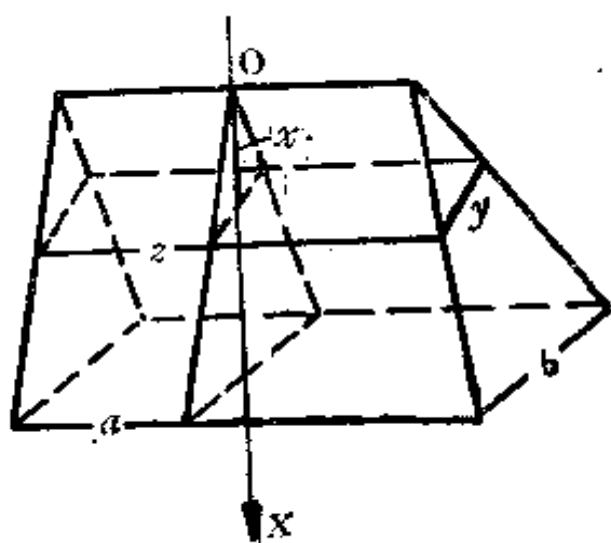


图 4.39

$$\frac{y}{b} = \frac{x}{h} \quad \text{或} \quad y = \frac{b}{h}x,$$

$$\frac{z-c}{a-c} = \frac{x}{h} \quad \text{或} \quad z = \frac{a-c}{h}x + c.$$

于是，所求顶楼的体积为

$$V = \int_0^h yz dx = \int_0^h \frac{b}{h} x \left(\frac{a-c}{h} x + c \right) dx$$

$$= \frac{b}{h} \cdot \frac{a-c}{h} \cdot \frac{1}{3} h^3 + \frac{bc}{h} \cdot \frac{1}{2} h^2$$

$$= \frac{bh}{6} (2a+c).$$

2457. 求截楔形的体积，其平行的上下底为边长分别等于 A , B 和 a , b 的矩形，而高等于 h 。

解 如图4.40所示,

$$OO' = \frac{A}{2},$$

$$QQ' = \frac{a}{2},$$

$$OQ = h.$$

设 $OP = x$, 则

$$PP' = \frac{a}{2} + \frac{h-x}{h} \left(\frac{A-a}{2} \right).$$

同样可得

$$LP' = \frac{b}{2} + \frac{h-x}{h} \left(\frac{B-b}{2} \right).$$

从而

$$\begin{aligned} \text{面积 } KLMN &= ab + (A-a)(B-b) \left(1 - \frac{x}{h} \right)^2 \\ &\quad + [a(B-b) + b(A-a)] \left(1 - \frac{x}{h} \right) \\ &= f(x). \end{aligned}$$

所求截楔形的体积为

$$V = \int_0^h f(x) dx = \frac{h}{6} [(2A+a)B + (2a+A)b].$$

2458. 求截锥体的体积, 其上下底为半轴长分别等于 A , B 和 a , b 的椭圆, 而高等于 h .

解 同2457题, 任一平行于上下底且距离下底为 x 的截面为一椭圆, 其半轴分别为

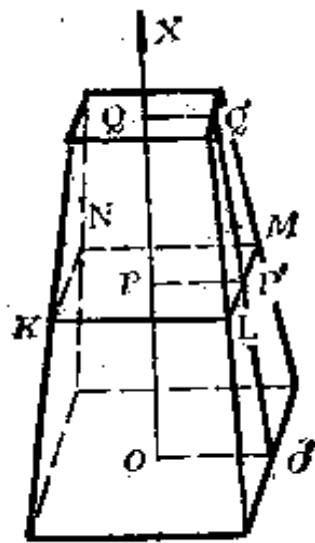


图 4.40

$$a' = a + \left(1 - \frac{x}{h}\right)(A - a)$$

及

$$b' = b + \left(1 - \frac{x}{h}\right)(B - b),$$

从而此截面的面积为

$$S(x) = \pi a' b'$$

$$= \pi \left\{ ab + (A - a)(B - b) \left(1 - \frac{x}{h}\right)^2 \right.$$

$$\left. + [a(B - b) + b(A - a)] \left(1 - \frac{x}{h}\right) \right\}.$$

所求的体积为

$$V = \int_0^h S(x) dx = \frac{\pi h}{6} [(2A + a)B + (A + 2a)b].$$

2459. 求旋转抛物体的体积，其底为 S ，而高等于 H 。

解 不失一般性，假设抛物线方程为

$$y^2 = 2px,$$

绕 Ox 轴旋转，如图 4.41 所示。

记 $OA = H$,

$OB = x$ ，按假设有

$$S = \pi \cdot \overline{AC}^2$$

$$= \pi(2pH)$$

$$= 2\pi pH,$$

距原点为 x 的截面面

积为

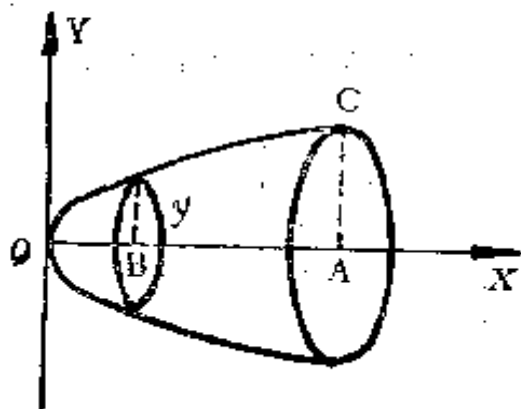


图 4.41

$$S(x) = \pi y^2 = 2\pi px,$$

于是, 所求的体积为

$$V = \int_0^H S(x) dx = \pi p H^2 = \frac{SH}{2}.$$

2460. 设立体之垂直于 Ox 轴的横截面的面积 $S = S(x)$, 依下面的二次式规律变化:

$$S(x) = Ax^2 + Bx + C \quad [a \leq x \leq b],$$

其中 A, B 及 C 为常数.

证明此物体之体积等于

$$V = \frac{H}{6} \left[S(a) + 4S\left(\frac{a+b}{2}\right) + S(b) \right],$$

其中 $H = b - a$ (辛普森公式).

$$\begin{aligned} \text{证} \quad V &= \int_a^b (Ax^2 + Bx + C) dx \\ &= \frac{A}{3}(b^3 - a^3) + \frac{B}{2}(b^2 - a^2) + C(b - a) \\ &= \frac{b-a}{6} [2A(b^2 + ab + a^2) + 3B(a+b) + 6C] \\ &= \frac{H}{6} [(Aa^2 + Ba + C) + (Ab^2 + Bb + C) \\ &\quad + A(a^2 + 2ab + b^2) + 2B(a+b) + 4C] \\ &= \frac{H}{6} \left[S(a) + S(b) + 4S\left(\frac{a+b}{2}\right) \right]. \end{aligned}$$

2461. 物体是点 $M(x, y, z)$ 的集合, 其中 $0 \leq z \leq 1$, 而且若 z 为有理数时, $0 \leq x \leq 1, 0 \leq y \leq 1$; 若 z 为

无理数时, $-1 \leq x \leq 0$, $-1 \leq y \leq 0$. 证明虽然对应的积分为

$$\int_0^1 S(z) dz = 1,$$

但此物体的体积不存在.

证 显然, 对任何 $0 \leq z \leq 1$, 不论 z 是有理数还是无理数, 都有 $S(z) = 1$. 从而

$$\int_0^1 S(z) dz = \int_0^1 dz = 1.$$

下证此物体 (V) 的体积不存在. 显然, 无完全含于 (V) 内的多面体 (X) 存在, 从而这种 (X) 的体积的上确界为零, 即 (V) 的内体积 $V_* = \sup \{X\} = 0$. 另一方面, (V) 的外体积 $V^* = \inf \{Y\}$, 其中的下确界是对所有完全包含着 (V) 的多面体 (Y) 的体积 Y 来取的. 由于 $0 \leq z \leq 1$ 中的有理数和无理数都在 $0 \leq z \leq 1$ 中是稠密的, 故, 显然, 上述任何完全包含着 (V) 的多面体 (Y) 都必完全包含着点集 $(Y_0) = \{(x, y, z) \mid 0 \leq z \leq 1; 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ 以及 } -1 \leq x \leq 0, -1 \leq y \leq 0\}$. 而 (Y_0) 又完全包含着 (V) , 并且 (Y_0) 的体积 $Y_0 = 2$. 由此可知 $V^* = \inf \{Y\} = 2$. 于是 $V_* \neq V^*$. 故 (V) 的体积不存在.

求下列曲面所围成的体积:

$$2462. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = \frac{c}{a}x, \quad z = 0.$$

解 如图 4.42 所示. 用垂直 Oy 轴的平面截割, 得一

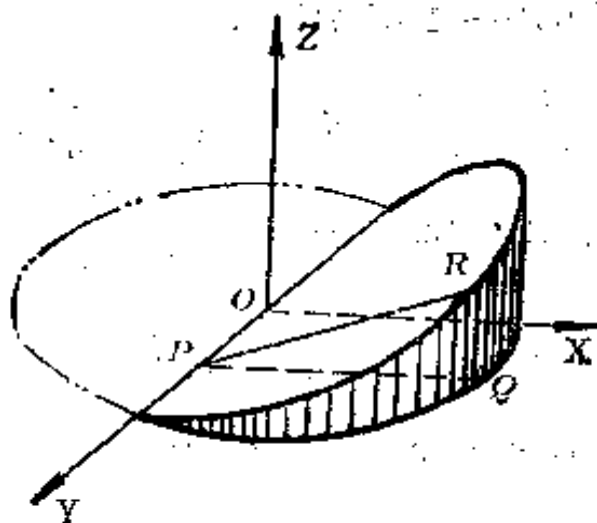


图 1.42

直角三角形PQR.

设 $OP=y$, 则高 $QR=\frac{c}{a}x$, 从而它的面积为

$$\frac{1}{2} \cdot \frac{c}{a} x^2 = \frac{ac}{2} \left(1 - \frac{y^2}{b^2}\right).$$

于是, 所求体积为

$$V = 2 \int_0^b \frac{ac}{2} \left(1 - \frac{y^2}{b^2}\right) dy = \frac{2}{3} abc.$$

2463. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (椭球).

解 用垂直于 Ox 轴的平面截椭球得截痕为一椭圆, 它在 yo_1z 平面上的投影为

$$\frac{y^2}{b^2 \left(1 - \frac{x^2}{a^2}\right)} + \frac{z^2}{c^2 \left(1 - \frac{x^2}{a^2}\right)} = 1.$$

由此显见其半轴分别为

$$b \cdot \sqrt{1 - \frac{x^2}{a^2}} \quad \text{及} \quad c \cdot \sqrt{1 - \frac{x^2}{a^2}},$$

从而此椭圆的面积为

$$S(x) = \pi bc \left(1 - \frac{x^2}{a^2} \right).$$

于是, 所求的椭球的体积为

$$\begin{aligned} V &= \int_{-a}^a S(x) dx = \int_{-a}^a \left(1 - \frac{x^2}{a^2} \right) \pi bc dx \\ &= \frac{4}{3} \pi abc. \end{aligned}$$

2464. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad z = \pm c.$

解 方程的图形为单叶双曲面, 用平面 $z = h$ 截得椭圆

$$\frac{x^2}{a^2 \left(1 + \frac{h^2}{c^2} \right)} + \frac{y^2}{b^2 \left(1 + \frac{h^2}{c^2} \right)} = 1,$$

其面积为

$$S(x) = \pi ab \left(1 + \frac{h^2}{c^2} \right).$$

于是, 所求的体积为

$$V = \pi ab \int_{-c}^c \left(1 + \frac{h^2}{c^2} \right) dh = \frac{8}{3} \pi abc.$$

2465. $x^2 + z^2 = a^2, y^2 + z^2 = a^2.$

解 如图4.43所示, 过点 $M(0, 0, z)$ 垂直于 Oz 轴作一平面, 在所给立体上截出一正方形, 其边长为 $\sqrt{a^2 - z^2}$, 所以其面积为

$$S(z) = a^2 - z^2,$$

于是, 所求的体积为

$$V = 8 \int_0^a (a^2 - z^2) dz = \frac{16}{3} a^3.$$

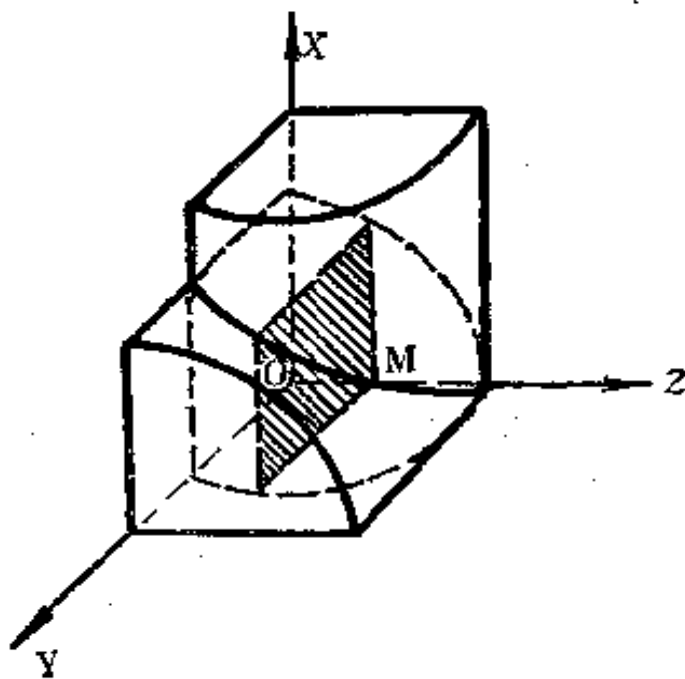


图 4.43

2466. $x^2 + y^2 + z^2 = a^2, x^2 + y^2 = ax.$

解 如图4.44所示, 过点 $M(x, 0, 0)$ 垂直于 Ox 轴作一平面, 在所给立体上截出一曲边梯形, 其曲边由方程

$$z = \sqrt{(a^2 - x^2) - y^2}$$

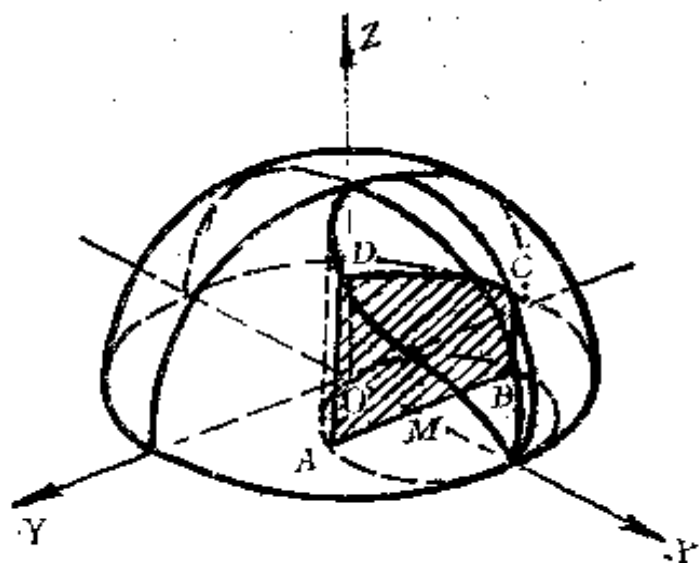


图 4.44

给出 (上半面),
其变化范围为:

$$-\sqrt{ax-x^2} \leq y \leq \sqrt{ax-x^2} \quad (\text{如图中 } ABCD).$$

从而其截面积为

$$\begin{aligned} S(x) &= 2 \int_0^{\sqrt{ax-x^2}} \sqrt{(a^2-x^2)-y^2} dy \\ &= a^{\frac{3}{2}} x^{\frac{1}{2}} - a^{\frac{1}{2}} x^{\frac{3}{2}} + (a^2-x^2) \arcsin \sqrt{\frac{x}{a+x}}. \end{aligned}$$

于是, 所求的体积为

$$\begin{aligned} V &= 2 \int_0^a S(x) dx \\ &= 2 \int_0^a \left\{ a^{\frac{3}{2}} x^{\frac{1}{2}} - a^{\frac{1}{2}} x^{\frac{3}{2}} \right. \\ &\quad \left. + (a^2-x^2) \arcsin \sqrt{\frac{x}{a+x}} \right\} dx \end{aligned}$$

$$\begin{aligned}
&= 4 \left\{ \frac{1}{3} a^3 - \frac{1}{5} a^3 + \left[\left(\frac{\pi a^3}{4} - \frac{1}{2} a^3 \right) \right. \right. \\
&\quad \left. \left. - \left(\frac{1}{12} \pi a^3 - \frac{13}{90} a^3 \right) \right] \right\} \\
&= \frac{2}{3} a^3 \left(\pi - \frac{4}{3} \right).
\end{aligned}$$

2467. $z^2 = b(a-x), x^2 + y^2 = ax.$

解 先求体积的四分之一部分, 截面积为

$$\begin{aligned}
S(x) &= \int_0^{\sqrt{ax-x^2}} \sqrt{b(a-x)} dy \\
&= \sqrt{ax-x^2} \cdot \sqrt{b(a-x)}.
\end{aligned}$$

从而

$$\begin{aligned}
\frac{1}{4} V &= \int_0^a S(x) dx = \int_0^a \sqrt{ax-x^2} \cdot \sqrt{b(a-x)} dx \\
&= \sqrt{b} \int_0^a \sqrt{x} (a-x) dx \\
&= \frac{4}{15} a^2 \sqrt{ab}.
\end{aligned}$$

于是, 所求的体积为

$$V = \frac{16}{15} a^2 \sqrt{ab}.$$

2468. $\frac{x^2}{a^2} + \frac{y^2}{z^2} = 1 (0 \leq z \leq a).$

解 固定 z , 则截面为一椭圆, 其面积为

$$P(z) = \pi a z.$$

于是, 所求的体积为

$$V = \int_0^a P(z) dz = \pi a \int_0^a z dz = \frac{\pi a^3}{2}.$$

2469⁺. $x + y + z^2 = 1$, $x = 0$, $y = 0$, $z = 0$.

解 固定 z , 则截面为一直角三角形, 其面积为

$$P(z) = \frac{1}{2}(1 - z^2)^2.$$

故所求体积

$$\begin{aligned} V &= \int_0^1 \frac{1}{2}(1 - z^2)^2 dz \\ &= \frac{1}{2} \int_0^1 (1 - 2z^2 + z^4) dz = \frac{4}{15}. \end{aligned}$$

注意, 曲面 $x + y + z^2 = 1$ 关于平面 $z = 0$ 对称, 故它与三个平面 $x = 0$, $y = 0$, $z = 0$ 围成的图形有两个, 一个位于 Oxy 平面之上, 一个位于 Oxy 平面之下, 彼此是对称的 (关于 Oxy 平面), 从而它们的体积相等. 我们以上求的是位于 Oxy 平面之上的那一个图形的体积.

2470. $x^2 + y^2 + z^2 + xy + yz + zx = a^2$.

解 不妨设 $a > 0$. 此为一有心椭球面. 固定 z , 得在平面 xoy 上的投影为

$$x^2 + xy + y^2 + zx + zy + (z^2 - a^2) = 0,$$

此截面的面积为

$$S(z) = -\frac{\pi \Delta}{\left(1 - \frac{1}{4}\right)^{\frac{3}{2}}} = -\frac{8\pi \Delta}{3\sqrt{3}},$$

其中

$$\Delta = \begin{vmatrix} 1 & \frac{1}{2} & \frac{z}{2} \\ \frac{1}{2} & 1 & \frac{z}{2} \\ \frac{z}{2} & \frac{z}{2} & z^2 - a^2 \end{vmatrix} = \frac{2z^2 - 3a^2}{4},$$

所以

$$S(z) = \frac{2(3a^2 - 2z^2)\pi}{3\sqrt{3}}.$$

z 的变化范围为适合下述不等式的集合:

$$2z^2 - 3a^2 \leq 0,$$

即

$$|z| \leq \sqrt{\frac{3}{2}}a.$$

于是, 所求的体积为

$$V = \int_{-\sqrt{\frac{3}{2}}|a|}^{\sqrt{\frac{3}{2}}|a|} \frac{2(3a^2 - 2z^2)\pi}{3\sqrt{3}} dz = \frac{4\sqrt{2}\pi}{3}a^3.$$

*) 此公式详见 Г. М. 菲赫金哥尔茨著《微积分学教程》第二卷第一分册第330目7.

2471. 证明: 将面积

$$a \leq x \leq b, \quad 0 \leq y \leq y(x),$$

(式中 $y(x)$ 为连续函数) 绕 Oy 轴旋转所成的旋转体体积等于

$$V_s = 2\pi \int_a^b xy(x) \cdot dx.$$

$$\begin{aligned}\text{证 } \Delta V_x &= \pi[(x + \Delta x)^2 - x^2]y(x) \\ &= 2\pi xy(x)\Delta x.\end{aligned}$$

于是, 所求的体积为

$$V_x = 2\pi \int_a^b xy(x)dx.$$

求下列曲线旋转所成曲面包围的体积:

2472. $y = b\left(\frac{x}{a}\right)^{\frac{3}{4}}$ ($0 \leq x \leq a$) 绕 Ox 轴 (半三次抛物线)。

解 所求的体积为

$$V_x = \pi b^2 \int_0^a \left(\frac{x}{a}\right)^{\frac{3}{2}} dx = \frac{3}{7}\pi ab^2.$$

2473. $y = 2x - x^2$, $y = 0$; (a) 绕 Ox 轴; (b) 绕 Oy 轴.

解 令 $y = 0$ 得 $x = 0$ 或 $x = 2$.

于是, 所求的体积为

$$(a) V_x = \pi \int_0^2 (2x - x^2)^2 dx = \frac{16\pi}{15},$$

$$(b) V_y = 2\pi \int_0^2 x(2x - x^2) dx = \frac{8\pi}{3}.$$

2474. $y = \sin x$, $y = 0$ ($0 \leq x \leq \pi$); (a) 绕 Ox 轴; (b) 绕 Oy 轴.

解 所求的体积为

$$(a) V_x = \pi \int_0^\pi \sin^2 x dx = \frac{\pi^2}{2};$$

$$(b) V_y = 2\pi \int_0^\pi x \sin x dx = 2\pi^2.$$

2475. $y = b\left(\frac{x}{a}\right)^2$; $y = b\left|\frac{x}{a}\right|$; (a) 绕 Ox 轴; (b) 绕 Oy 轴.

解 交点为 (a, b) 及 $(-a, b)$.

所求的体积为

$$\begin{aligned} \text{(a)} \quad V_x &= 2\pi \int_0^a \left(b^2 \frac{x^2}{a^2} - b^2 \frac{x^4}{a^4} \right) dx \\ &= \frac{4\pi}{15} ab^3; \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad V_y &= \pi \int_0^b \left(\frac{a^2 y}{b} - \frac{a^2 y^2}{b^2} \right) dy \\ &= \frac{\pi a^2 b}{6}. \end{aligned}$$

2476. $y = e^{-x}$, $y = 0$ ($0 \leq x < +\infty$); (a) 绕 Ox 轴; (b) 绕 Oy 轴.

解 所求的体积为

$$\text{(a)} \quad V_x = \pi \int_0^{+\infty} e^{-2x} dx = \frac{\pi}{2};$$

$$\text{(b)} \quad V_y = \pi \int_0^1 (-\ln y)^2 dy = 2\pi.$$

2477. $x^2 + (y-b)^2 = a^2$ ($0 < a \leq b$) 绕 Ox 轴.

解 $y_1 = b + \sqrt{a^2 - x^2}$, $y_2 = b - \sqrt{a^2 - x^2}$
($-a \leq x \leq a$).

所求的体积为

$$V_x = \pi \int_{-a}^a (y_1^2 - y_2^2) dx$$

$$= 8b\pi \int_0^a \sqrt{a^2 - x^2} dx = 2\pi^2 a^2 b.$$

2478. $x^2 - xy + y^2 = a^2$ 绕 Ox 轴.

解 原方程即 $y^2 - xy + x^2 - a^2 = 0$, 从而

$$y = \frac{x \pm \sqrt{4a^2 - 3x^2}}{2},$$

函数的定义域为 $\left[-\frac{2}{\sqrt{3}}a, \frac{2}{\sqrt{3}}a\right]$. 与 Ox 轴的交

点分别为 $x = -a$ 与 $x = a$.

于是, 所求的体积为

$$\begin{aligned} V_x &= 2 \left\{ \pi \int_0^a \frac{1}{4} \left(x + \sqrt{4a^2 - 3x^2} \right)^2 dx \right. \\ &\quad \left. + \pi \int_0^{\frac{2}{\sqrt{3}}a} \left[\frac{1}{4} \left(x + \sqrt{4a^2 - 3x^2} \right)^2 \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \left(x - \sqrt{4a^2 - 3x^2} \right)^2 \right] dx \right\} \\ &= \frac{\pi}{2} \int_0^a (4a^2 - 2x^2 + 2x\sqrt{4a^2 - 3x^2}) dx \\ &\quad + 2\pi \int_0^{\frac{2}{\sqrt{3}}a} x\sqrt{4a^2 - 3x^2} dx \\ &= \pi \left[2a^3 - \frac{1}{3}a^3 - \frac{1}{9}(4a^2 - 3x^2)^{\frac{3}{2}} \right]_0^a \\ &\quad - \frac{2}{9}(4a^2 - 3x^2)^{\frac{3}{2}} \Big|_0^{\frac{2}{\sqrt{3}}a} = \frac{8}{3}\pi a^3. \end{aligned}$$

2479. $y = e^{-x} \sqrt{\sin x}$ ($0 \leq x < +\infty$) 绕 Ox 轴.

解 函数定义域为 $[2n\pi, (2n+1)\pi]$, ($n=0, 1, 2, \dots$), 故所求的体积为

$$\begin{aligned} V_x &= \pi \sum_{n=0}^{\infty} \int_{2n\pi}^{(2n+1)\pi} e^{-2x} \sin x dx \\ &= \sum_{n=0}^{\infty} \frac{\pi}{5} e^{-2x} (-2\sin x - \cos x) \Big|_{2n\pi}^{(2n+1)\pi} \\ &= \frac{\pi}{5} (e^{-2\pi} + 1) \sum_{n=0}^{\infty} e^{-4n\pi} \\ &= \frac{\pi}{5} \cdot \frac{e^{-2\pi} + 1}{1 - e^{-4\pi}} = \frac{\pi}{5(1 - e^{-2\pi})}. \end{aligned}$$

2480. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$),
 $y = 0$;

(a) 绕 Ox 轴; (b) 绕 Oy 轴; (B) 绕直线 $y = 2a$.

解 所求的体积为

$$(a) V_x = \pi \int_0^{2\pi} a^3 (1 - \cos t)^3 dt = 5\pi^2 a^3;$$

$$\begin{aligned} (b) V_y &= 2\pi \int_0^{2\pi} a^3 (t - \sin t) (1 - \cos t)^2 dt \\ &= 6\pi^3 a^3; \end{aligned}$$

(B) 作平移: $y = \bar{y} + 2a$, $x = \bar{x}$, 则曲线方程为

$$\begin{aligned} \bar{x} &= a(t - \sin t), \quad \bar{y} = -a(1 + \cos t), \quad \text{及} \\ \bar{y} &= -2a. \end{aligned}$$

于是, 所求的体积为

$$V_x = \pi \int_0^{2\pi} [4a^2 - a^2(1 + \cos t)^2] a(1 - \cos t) dt$$

$$= 7\pi^2 a^3.$$

2481. $x = a \sin^3 t$, $y = b \cos^3 t$ ($0 \leq t \leq 2\pi$);

(a) 绕 Ox 轴; (b) 绕 Oy 轴.

解 所求的体积为

$$(a) V_x = 2\pi \int_0^{\frac{\pi}{2}} (b^2 \cos^6 t)(3a \sin^2 t \cos t) dt$$

$$= 6\pi ab^2 \left(\int_0^{\frac{\pi}{2}} \cos^7 t dt - \int_0^{\frac{\pi}{2}} \cos^5 t dt \right)$$

$$= 6\pi ab^2 \left(\frac{6!!}{7!!} - \frac{8!!}{9!!} \right)^*$$

$$= \frac{32}{105} \pi ab^2;$$

(b) 利用对称性, 只须将上述答案中 a, b 对调即得

$$V_y = \frac{32}{105} \pi a^2 b.$$

*) 利用2282题的结果.

2482. 证明把面积

$$0 \leq \alpha \leq \varphi \leq \beta \leq \pi, \quad 0 \leq r \leq r(\varphi)$$

(φ 与 r 为极坐标) 绕极轴旋转所成的体积等于

$$V = \frac{2\pi}{3} \int_{\alpha}^{\beta} r^3(\varphi) \sin \varphi d\varphi.$$

证 证法一:

微面积 $dS = r d\varphi dr$ 绕极轴旋转所得微环形体积

$$dV = 2\pi r \sin\varphi dS = 2\pi r^2 \sin\varphi d\varphi dr.$$

于是, 所求的体积

$$\begin{aligned} V &= 2\pi \int_a^b \sin\varphi d\varphi \int_a^{r(\varphi)} r^2 dr \\ &= \frac{2\pi}{3} \int_a^b r^3(\varphi) \sin\varphi d\varphi. \end{aligned}$$

证法二:

应用直角坐标系下的古尔金第二定理*) 来证明. 对于微小面积元, 它的重心可以看成在点 $(\frac{2}{3} r \cos\varphi,$

$\frac{2}{3} r \sin\varphi)$ 处 (图

4.45).

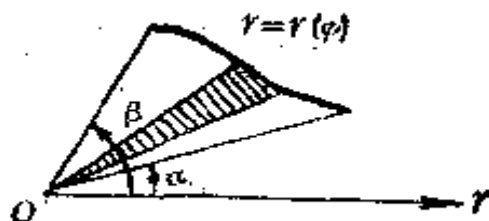


图 4.45

于是面积元素 $dS = \frac{1}{2} r^2 d\varphi$, 其所对应的绕极轴旋转所成的体积元素为

$$dV = 2\pi \frac{2}{3} r \sin\varphi \cdot \frac{1}{2} r^2 d\varphi.$$

所以

$$V = \frac{2\pi}{3} \int_a^b r^3(\varphi) \sin\varphi d\varphi.$$

*) 参看2506题.

求下列由极坐标所表出的面积经旋转后所得的体积:

2483. $r = a(1 + \cos \varphi)$ ($0 \leq \varphi \leq 2\pi$):

(a) 绕极轴; (6) 绕直线 $r \cos \varphi = -\frac{a}{4}$.

$$\begin{aligned} \text{解 (a) } V &= \frac{2\pi}{3} \int_0^\pi a^3 (1 + \cos \varphi)^3 \sin \varphi d\varphi \\ &= \frac{8\pi a^3}{3}; \end{aligned}$$

(6) 方法一:

所求的旋转体的体积为

$$\begin{aligned} V &= 2\pi \int_0^\pi r^2 \left(\frac{2}{3} r \cos \varphi + \frac{a}{4} \right) d\varphi \\ &= \frac{4\pi a^3}{3} \int_0^\pi (1 + \cos \varphi)^2 \cos \varphi d\varphi \\ &\quad + \frac{\pi a^3}{2} \int_0^\pi (1 + \cos \varphi)^2 d\varphi \\ &= \left(4\pi a^3 + \frac{\pi a^3}{2} \right) \int_0^\pi \cos^2 \varphi d\varphi \\ &\quad + \frac{4\pi a^3}{3} \int_0^\pi \cos^3 \varphi d\varphi + \frac{\pi^2 a^3}{2} \\ &= \left(4\pi a^3 + \frac{\pi a^3}{2} \right) \frac{\pi}{2} + \frac{4\pi a^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \pi + \frac{\pi^2 a^3}{2} \\ &= \frac{13}{4} \pi^2 a^3. \end{aligned}$$

注 (1) 在 V 的表达式中 $\frac{2}{3} r \cos \varphi$ 的系数 $\frac{2}{3}$ 是把微小

面积集中在其重心 $(\frac{2}{3}r, \varphi)$ 处得出的。

$$(2) \int_0^{\pi} \cos^{2k+1} \varphi d\varphi = 0,$$

$$\int_0^{\pi} \cos^{2k} \varphi d\varphi = \frac{(2k-1)(2k-3)\cdots 3 \cdot 1}{2k(2k-2)\cdots 4 \cdot 2} \pi.$$

方法二:

心脏线 $r = a(1 + \cos \varphi)$ 的面积为 $\frac{3\pi a^2}{2}$ ^{*}, 而其

重心为 $\varphi_0 = 0, r_0 = \frac{5}{6}a$ ^{**}。根据古尔金第二定理可得所求的体积为

$$V = 2\pi \left(\frac{5a}{6} + \frac{a}{4} \right) \frac{3\pi a^2}{2} = \frac{13}{4}\pi^2 a^3.$$

*) 利用2419题的结果。

**) 利用2512题的结果。

2484. $(x^2 + y^2)^2 = a^2(x^2 - y^2);$

(a) 绕 Ox 轴; (b) 绕 Oy 轴; (B) 绕直线 $y = x$ 。

解 (a) 曲线的极坐标方程为

$$r^2 = a^2(2\cos^2 \varphi - 1).$$

$$V_x = 2 \cdot \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} [a^2(2\cos^2 \varphi - 1)]^{\frac{3}{2}} \sin \varphi d\varphi.$$

由于

$$\int (2\cos^2 \varphi - 1)^{\frac{3}{2}} \sin \varphi d\varphi$$

$$\begin{aligned}
&= \int ((\sqrt{2} \cos \varphi)^2 - 1)^{\frac{3}{2}} d(\sqrt{2} \cos \varphi) \cdot \left(-\frac{1}{\sqrt{2}}\right) \\
&= -\frac{1}{\sqrt{2}} \left[\frac{\sqrt{2} \cos \varphi}{8} (4 \cos^2 \varphi \right. \\
&\quad \left. - 5) \sqrt{2 \cos^2 \varphi - 1} + \frac{3}{8} \ln (\sqrt{2} \cos \varphi \right. \\
&\quad \left. + \sqrt{2 \cos^2 \varphi - 1}) \right] + C.
\end{aligned}$$

所以

$$\begin{aligned}
V_x &= \frac{4\pi a^3}{3\sqrt{2}} \left[\frac{\sqrt{2} \cos \varphi}{8} (4 \cos^2 \varphi \right. \\
&\quad \left. - 5) \sqrt{2 \cos^2 \varphi - 1} + \frac{3}{8} \ln (\sqrt{2} \cos \varphi \right. \\
&\quad \left. + \sqrt{2 \cos^2 \varphi - 1}) \right] \Big|_0^{\frac{\pi}{4}} \\
&= \frac{4\pi a^3}{3\sqrt{2}} \left[\frac{3}{8} \ln (\sqrt{2} + 1) - \frac{\sqrt{2}}{8} \right] \\
&= \frac{1}{4} \pi a^3 \left[\sqrt{2} \ln (\sqrt{2} + 1) - \frac{2}{3} \right].
\end{aligned}$$

(6) 利用对称性知, 所求的体积为

$$\begin{aligned}
V_v &= \frac{4\pi}{3} \int_0^{\frac{\pi}{4}} r^3 \cos \varphi d\varphi \\
&= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} \sqrt{\cos^3 2\varphi} \cos \varphi d\varphi.
\end{aligned}$$

令 $\sin \varphi = \frac{1}{\sqrt{2}} \sin x$, 则 $\sqrt{\cos 2\varphi} = \cos x$,

$\cos \varphi d\varphi = \frac{1}{\sqrt{2}} \cos x dx$, 并且 x 的变化范围为

$(0, \frac{\pi}{2})$. 于是, 得

$$V = \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}} \cos^4 x dx$$

$$= \frac{4\pi a^3}{3} \cdot \frac{1}{\sqrt{2}} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2 a^3}{4\sqrt{2}}.$$

(B) 利用对称性知所求的体积为

$$V = \frac{4\pi}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3 \sin\left(\frac{\pi}{4} - \varphi\right) d\varphi$$

$$= \frac{4\pi a^3}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos^3 2\varphi} \left(-\frac{1}{\sqrt{2}} \cos \varphi\right.$$

$$\left. - \frac{1}{\sqrt{2}} \sin \varphi\right) d\varphi$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos^3 2\varphi} \cos \varphi d\varphi.$$

若用本题 (6) 的变换, 即得

$$V = \frac{4\pi a^3}{3\sqrt{2}} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}} \cos^4 x dx$$

$$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}} \cos^4 x dx$$

$$= \frac{4\pi a^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4}.$$

2485. 求绕极轴把面积

$$a \leq r \leq a \sqrt{2 \sin 2\varphi}$$

旋转而成的旋转体体积.

解 $r = a$ 与 $r = a \sqrt{2 \sin 2\varphi}$, 在第一象限部分的交

点的极角分别为 $\alpha = \frac{\pi}{12}$ 及 $\beta = \frac{5\pi}{12}$. 利用对称性知,

所求的体积应为

$$\begin{aligned} V &= \frac{4\pi}{3} \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} [(a \sqrt{2 \sin 2\varphi})^3 - a^3] \sin \varphi d\varphi \\ &= \frac{4\pi a^3}{3} \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (4\sqrt{2} \sqrt{\sin 2\varphi} \sin^2 \varphi \cos \varphi \\ &\quad - \sin \varphi) d\varphi. \end{aligned}$$

为求上述积分, 令

$$I_1 = \int \sqrt{\sin 2\varphi} \sin^2 \varphi \cos \varphi d\varphi,$$

$$I_2 = \int \sqrt{\sin 2\varphi} \cos^2 \varphi \cos \varphi d\varphi,$$

$$\text{则 } I_2 - I_1 = \frac{1}{3} \cos \varphi (\sin 2\varphi)^{\frac{3}{2}} + \frac{2}{3} I_1$$

即

$$I_2 - \frac{5}{3} I_1 = \frac{1}{3} \cos \varphi \cdot (\sin 2\varphi)^{\frac{3}{2}}. \quad (1)$$

又

$$\begin{aligned} I_2 + I_1 &= \int \sqrt{\sin 2\varphi} \cos \varphi d\varphi \\ &= \sqrt{2} \int \frac{\operatorname{tg} \varphi}{1 + \operatorname{tg}^2 \varphi} \sqrt{\operatorname{ctg} \varphi} d\varphi. \end{aligned}$$

令 $\operatorname{tg} \varphi = t$ ，就可将上述积分化成二项式的微分的积分。积分之，得

$$\begin{aligned} I_2 + I_1 &= \frac{1}{2} \sin \varphi \sqrt{\sin 2\varphi} + \frac{1}{2} \ln(\sin \varphi + \cos \varphi \\ &\quad - \sqrt{\sin 2\varphi}) + \frac{1}{4} [\ln(\sin \varphi + \cos \varphi \\ &\quad + \sqrt{\sin 2\varphi}) + \operatorname{arc} \sin(\sin \varphi \\ &\quad - \cos \varphi)]. \end{aligned} \quad (2)$$

(2) - (1)，得

$$\begin{aligned} I_1 &= \frac{3}{8} \left\{ \frac{1}{2} \sin \varphi \sqrt{\sin 2\varphi} + \frac{1}{2} \ln(\sin \varphi + \cos \varphi \right. \\ &\quad - \sqrt{\sin 2\varphi}) + \frac{1}{4} [\ln(\sin \varphi + \cos \varphi \\ &\quad + \sqrt{\sin 2\varphi}) + \operatorname{arc} \sin(\sin \varphi - \cos \varphi)] \\ &\quad \left. - \frac{1}{3} \cos \varphi \cdot (\sin 2\varphi)^{\frac{3}{2}} \right\} + C. \end{aligned}$$

从而，得

$$\begin{aligned} &\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \sqrt{\sin 2\varphi} \sin^2 \varphi \cos \varphi d\varphi \\ &= \frac{1}{8} + \frac{3}{64}\pi. \end{aligned}$$

因此, 所求的体积为

$$\begin{aligned}
 V &= \frac{4\pi a^3}{3} \left[4\sqrt{2} \left(\frac{1}{8} + \frac{3\pi}{64} \right) + \cos\varphi \left| \frac{\frac{8\pi}{12}}{\frac{x}{12}} \right| \right] \\
 &= \frac{\pi^2 a^3}{2\sqrt{2}}.
 \end{aligned}$$

§8. 旋转曲面表面积的计算法

平滑的曲线 AB 绕 Ox 轴旋转所成曲面的面积等于

$$P = 2\pi \int_A^B y ds,$$

式中 ds 为弧的微分,

求旋转下列曲线所成曲面的面积:

2486. $y = x\sqrt{\frac{x}{a}}$ ($0 \leq x \leq a$) 绕 Ox 轴.

解 $\sqrt{1+y'^2} = \sqrt{1 + \frac{9x}{4a}}$.

于是, 所求的表面积为

$$\begin{aligned}
 P_s &= 2\pi \int_0^a x \sqrt{\frac{x}{a}} \sqrt{1 + \frac{9x}{4a}} dx \\
 &= \frac{3\pi}{a} \int_0^a x \sqrt{x^2 + \frac{4ax}{9}} dx \\
 &= \frac{3\pi}{a} \int_0^a \left(x + \frac{2a}{9} \right) \sqrt{\left(x + \frac{2a}{9} \right)^2 - \left(\frac{2a}{9} \right)^2} dx
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2a}{9}) - \frac{3\pi}{a} \cdot \frac{2a}{9} \int_0^a \sqrt{x^2 + \frac{4ax}{9}} dx \\
& = \frac{3\pi}{a} \frac{1}{3} \left(x^2 + \frac{4ax}{9}\right)^{\frac{3}{2}} \Big|_0^a \\
& \quad - \frac{2\pi}{3} \left\{ \frac{x + \frac{2a}{9}}{2} \sqrt{x^2 + \frac{4ax}{9}} \right. \\
& \quad \left. - \frac{4a^2}{81} \ln \left(x + \frac{2a}{9} + \sqrt{x^2 + \frac{4ax}{9}} \right) \right\} \Big|_0^a \\
& = \frac{13\sqrt{13}}{27} \pi a^2 - \frac{11\sqrt{13}}{81} \pi a^2 \\
& \quad + \frac{4\pi a^2}{243} \ln \frac{11 + 3\sqrt{13}}{2} \\
& = \frac{4\pi a^2}{243} \left(21\sqrt{13} + 2 \ln \frac{11 + 3\sqrt{13}}{2} \right).
\end{aligned}$$

2487. $y = a \cos \frac{\pi x}{2b}$ ($|x| \leq b$) 绕 Ox 轴.

解 $y' = -\frac{\pi a}{2b} \sin \frac{\pi x}{2b}$,

$$\sqrt{1 + y'^2} = \frac{1}{2b} \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}}.$$

于是, 所求的表面积为

$$\begin{aligned}
P_s &= 2\pi \int_{-b}^b y \sqrt{1 + y'^2} dx \\
&= 2\pi \int_{-b}^b \frac{a}{2b} \cos \frac{\pi x}{2b} \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{4}{\pi} \left[\frac{1}{2} \pi a \sin \frac{\pi x}{2b} \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}} \right. \\
&\quad \left. + \frac{4b^2}{2} \ln \left| \pi a \sin \frac{\pi x}{2b} \right. \right. \\
&\quad \left. \left. + \sqrt{4b^2 + \pi^2 a^2 \sin^2 \frac{\pi x}{2b}} \right| \right] \Big|_0^{\frac{\pi}{2}} \\
&= 2a \sqrt{\pi^2 a^2 + 4b^2} \\
&\quad + \frac{8b^2}{\pi} \ln \frac{\pi a + \sqrt{4b^2 + \pi^2 a^2}}{2b}.
\end{aligned}$$

2488. $y = \operatorname{tg} x$ ($0 \leq x \leq \frac{\pi}{4}$) 绕 Ox 轴.

解 $\sqrt{1 + y'^2} = \sqrt{1 + \sec^4 x} = \frac{\sqrt{\cos^4 x + 1}}{\cos^2 x}$.

于是, 所求的表面积为

$$\begin{aligned}
P_s &= 2\pi \int_0^{\frac{\pi}{4}} \operatorname{tg} x \cdot \frac{\sqrt{\cos^4 x + 1}}{\cos^2 x} dx \\
&= \pi \int_0^{\frac{\pi}{4}} \sqrt{\cos^4 x + 1} d\left(\frac{1}{\cos^2 x}\right) \\
&= \pi \left[\frac{\sqrt{\cos^4 x + 1}}{\cos^2 x} - \ln(\cos^2 x \right. \\
&\quad \left. + \sqrt{\cos^4 x + 1}) \right] \Big|_0^{\frac{\pi}{4}} \\
&= \pi \left[\sqrt{5} - \sqrt{2} \right].
\end{aligned}$$

$$+ \ln \frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2} \Big].$$

2489. $y^2 = 2px$ ($0 \leq x \leq x_0$): (a) 绕 Ox 轴,
(b) 绕 Oy 轴.

解 (a) $\sqrt{1+y'^2} = \frac{\sqrt{p+2x}}{\sqrt{2x}}$.

于是, 所求的表面积为

$$\begin{aligned} P_x &= 2\pi \int_0^{x_0} \sqrt{2px} \cdot \frac{\sqrt{p+2x}}{\sqrt{2x}} dx \\ &= \frac{2\pi}{3} \left[(2x_0 + p) \sqrt{2px_0 + p^2} - p^2 \right]. \end{aligned}$$

(b) $\sqrt{1+x_y'^2} = \frac{\sqrt{p^2+y^2}}{p}$.

于是, 所求的表面积为

$$\begin{aligned} P_y &= 4\pi \int_0^{\sqrt{2px_0}} x \sqrt{1+x_y'^2} dy \\ &= 4\pi \int_0^{\sqrt{2px_0}} \frac{y^2}{2p} \cdot \frac{\sqrt{p^2+y^2}}{p} dy \\ &= \frac{2\pi}{p^2} \int_0^{\sqrt{2px_0}} y^2 \sqrt{p^2+y^2} dy \\ &= \frac{2\pi}{p^2} \left[\frac{y(2y^2+p^2)}{8} \sqrt{p^2+y^2} \right. \\ &\quad \left. - \frac{p^4}{8} \ln(y + \sqrt{y^2+p^2}) \right] \Big|_0^{\sqrt{2px_0}} \\ &= \frac{\pi}{4} \left[(p+4x_0) \sqrt{2x_0(p+2x_0)} \right. \end{aligned}$$

$$- p^2 \ln \frac{\sqrt{2x_0} + \sqrt{p+2x_0}}{\sqrt{p}} \Bigg\}.$$

2490. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($0 < b \leq a$): (a) 绕 Ox 轴;

(b) 绕 Oy 轴.

解 (a) $y^2 = b^2 - \frac{b^2}{a^2}x^2$, $yy' = -\frac{b^2}{a^2}x$,

$$\begin{aligned} y \sqrt{1+y'^2} &= \sqrt{y^2 + (yy')^2} \\ &= \frac{b}{a} \sqrt{a^2 - \frac{a^2 - b^2}{a^2}x^2} = \frac{b}{a} \sqrt{a^2 - e^2 x^2}. \end{aligned}$$

于是, 所求的表面积为

$$\begin{aligned} P_1 &= 2\pi \frac{b}{a} \int_{-a}^a \sqrt{a^2 - e^2 x^2} dx \\ &= \frac{2\pi b}{a} \left(a\sqrt{a^2 - e^2 a^2} + \frac{a^2}{e} \arcsin e \right) \\ &= 2\pi b \left(b + \frac{a}{e} \arcsin e \right), \end{aligned}$$

其中 $e = \frac{\sqrt{a^2 - b^2}}{a}$ 是椭圆之离心率.

(b) 将 x, y 轴对调, 即将 x 轴作为短轴. 于是在所得出的 $y \sqrt{1+y'^2}$ 中仅需将 a 与 b 的位置对调一下即可, 即

$$y \sqrt{1+y'^2} = \frac{a}{b} \sqrt{b^2 + \frac{a^2 - b^2}{b^2}x^2}$$

$$= \frac{a}{b} \sqrt{b^2 + \frac{c^2}{b^2} x^2}.$$

于是, 所求表面积为

$$\begin{aligned} P_s &= 2\pi \frac{a}{b} \int_{-b}^b \sqrt{b^2 + \frac{c^2}{b^2} x^2} dx \\ &= 2\pi a \frac{1}{b} \left[\frac{x}{2} \sqrt{b^2 + \frac{c^2}{b^2} x^2} \right. \\ &\quad \left. + \frac{b^3}{2c} \ln \left(\frac{c}{b} x + \sqrt{b^2 + \frac{c^2}{b^2} x^2} \right) \right] \Big|_{-b}^b \\ &= 2\pi a \left(\sqrt{b^2 + c^2} + \frac{b^2}{2c} \ln \left[\frac{\sqrt{b^2 + c^2} + c}{\sqrt{b^2 + c^2} - c} \right] \right) \\ &= 2\pi a \left(a + \frac{b^2}{2c} \ln \left[\frac{a+c}{a-c} \right] \right) \\ &= 2\pi a \left(a + \frac{b^2}{2a} \cdot \frac{1}{e} \ln \frac{1+\varepsilon}{1-\varepsilon} \right) \\ &= 2\pi a \left\{ a + \frac{b^2}{a} \cdot \frac{1}{e} \ln \left[\frac{a}{b} (1+\varepsilon) \right] \right\}. \end{aligned}$$

2491. $x^2 + (y-b)^2 = a^2$ ($b \geq a$) 绕 Ox 轴.

解 此圆分成两单值支

$$y = b + \sqrt{a^2 - x^2} \text{ 及 } y = b - \sqrt{a^2 - x^2}.$$

于是, 所求的表面积为

$$\begin{aligned} P_s &= 2\pi \int_{-a}^a (b + \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}} dx \\ &\quad + 2\pi \int_{-a}^a (b - \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}} dx \end{aligned}$$

$$= 4\pi^2 ab.$$

2492. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ 绕 Ox 轴.

解 $y' = -\sqrt[3]{\frac{y}{x}}, \sqrt{1+y'^2} = \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$

于是, 所求的表面积为

$$\begin{aligned} P_x &= 2 \cdot 2\pi \int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} dx \\ &= -\frac{12\pi a^{\frac{1}{3}}}{5} (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{5}{2}} \Big|_0^a \\ &= \frac{12\pi a^2}{5}. \end{aligned}$$

2493. $y = a \operatorname{ch} \frac{x}{a}$ ($|x| \leq b$): (a) 绕 Ox 轴;

(b) 绕 Oy 轴.

解 (a) $\sqrt{y'^2 + 1} = \sqrt{\operatorname{sh}^2 \frac{x}{a} + 1} = \operatorname{ch} \frac{x}{a}.$

于是, 所求的表面积为

$$\begin{aligned} P_x &= 2\pi a \int_{-b}^b \operatorname{ch}^2 \frac{x}{a} dx \\ &= 2\pi a \int_0^b \left(1 + \operatorname{ch} \frac{2x}{a}\right) dx \\ &= \pi a \left(2b + a \operatorname{sh} \frac{2b}{a}\right). \end{aligned}$$

$$\begin{aligned}
 (6) P_s &= 2\pi \int_0^b x \sqrt{1+y'^2} dx \\
 &= 2\pi \int_0^b x \operatorname{ch} \frac{x}{a} dx \\
 &= 2\pi a \left(a + b \operatorname{sh} \frac{b}{a} - a \operatorname{ch} \frac{b}{a} \right).
 \end{aligned}$$

2494. $\pm x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$ 绕 Ox 轴.

解 $x'_y = \mp \frac{\sqrt{a^2 - y^2}}{y}$, $\sqrt{1+x'^2} = \frac{a}{y}$

$(0 \leq y \leq a)$.

于是, 所求的表面积为

$$P_s = 2 \cdot 2\pi \int_0^a y \frac{a}{y} dy = 4\pi a^2.$$

2495. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$):

(a) 绕 Ox 轴; (b) 绕 Oy 轴; (c) 绕直线 $y = 2a$;

解 先求 ds :

$$ds = \sqrt{x_t'^2 + y_t'^2} dt = 2a \sin \frac{t}{2} dt.$$

于是, 所求的表面积为

$$(a) P_s = 2\pi \int_0^{2\pi} a(1 - \cos t) \cdot 2a \sin \frac{t}{2} dt$$

$$= 16\pi a^2 \int_0^\pi \sin^3 u du = \frac{64}{3}\pi a^2.$$

$$(b) P_s = 2\pi \int_0^{2\pi} a(t - \sin t) \cdot 2a \sin \frac{t}{2} dt$$

$$= 4\pi a^2 \int_0^{2\pi} (t - \sin t) \sin \frac{t}{2} dt = 16\pi^2 a^2.$$

(B) 作平移 $\bar{x} = x$, $\bar{y} = y + 2a$ 则

$$\bar{y} = -a(1 + \cos t),$$

于是, 所求的表面积为

$$\begin{aligned} P_{\bar{x}} &= \left| 2\pi \int_0^{2\pi} \left[-a(1 + \cos t) \right] 2a \sin \frac{t}{2} dt \right|^{*}) \\ &= \frac{32}{3}\pi a^2. \end{aligned}$$

*) 在此取绝对值, 是由于被积函数始终不为正之故.

2496. $x = a \cos^3 t$, $y = a \sin^3 t$ 绕直线 $y = x$.

解 先求 ds :

$$ds = \sqrt{x_1'^2 + y_1'^2} dt$$

$$= \begin{cases} 3a \sin t \cos t dt, & \text{当 } \frac{\pi}{4} \leq t \leq \frac{\pi}{2}, \\ -3a \sin t \cos t dt, & \text{当 } \frac{\pi}{2} \leq t \leq \frac{3\pi}{4}. \end{cases}$$

利用对称性, 并作旋转, 即得所求的表面积为

$$\begin{aligned} P &= 2 \left[2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{yx}{\sqrt{2}} \sqrt{x_1'^2 + y_1'^2} dt \right. \\ &\quad \left. + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{y-x}{\sqrt{2}} \sqrt{x_1'^2 + y_1'^2} dt \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{4\pi}{\sqrt{2}} \left[\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (a \sin^3 t - a \cos^3 t) \cdot 3a \sin t \cos t \, dt \right. \\
&\quad \left. - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (a \sin^3 t - a \cos^3 t) \cdot 3a \sin t \cos t \, dt \right] \\
&= \frac{12\pi a^2}{\sqrt{2}} \left[\left(\frac{1}{5} \sin^5 t + \frac{1}{5} \cos^5 t \right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right. \\
&\quad \left. - \left(\frac{1}{5} \sin^5 t + \frac{1}{5} \cos^5 t \right) \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \right] \\
&= \frac{3}{5} \pi a^2 (4\sqrt{2} - 1).
\end{aligned}$$

2497. $r = a(1 + \cos \varphi)$ 绕极轴.

解 $ds = \sqrt{r^2 + r'^2} \, d\varphi = 2a \cos \frac{\varphi}{2} \, d\varphi,$

$$\begin{aligned}
y &= r \sin \varphi = a(1 + \cos \varphi) \sin \varphi \\
&= 4a \cos^3 \frac{\varphi}{2} \sin \frac{\varphi}{2}.
\end{aligned}$$

于是, 所求的表面积为

$$P = 2\pi \int_0^{\pi} 8a^2 \cos^4 \frac{\varphi}{2} \sin \frac{\varphi}{2} \, d\varphi = \frac{32}{5} \pi a^2.$$

2498. $r^2 = a^2 \cos 2\varphi$: (a) 绕极轴; (b) 绕轴 $\varphi = \frac{\pi}{2}$;

(B) 绕轴 $\varphi = \frac{\pi}{4}$.

解 (a) $y = a \sqrt{\cos 2\varphi} \sin \varphi, ds = \frac{a}{\sqrt{\cos 2\varphi}} \, d\varphi.$

于是, 所求的表面积为

$$P = 2 \cdot 2\pi \int_0^{\frac{\pi}{4}} a^2 \sin \varphi \, d\varphi = 2\pi a^2 (2 - \sqrt{2}).$$

$$(6) \quad x = a\sqrt{\cos 2\varphi} \cos \varphi \quad \left(-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}\right).$$

于是, 所求的表面积为

$$\begin{aligned} P &= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\sqrt{\cos 2\varphi} \cos \varphi \frac{a}{\sqrt{\cos 2\varphi}} d\varphi \\ &= 2\pi a^2 \sqrt{2}. \end{aligned}$$

$$(B) \quad x = a\sqrt{\cos 2\varphi} \cos \varphi, \quad y = a\sqrt{\cos 2\varphi} \sin \varphi,$$

$$ds = \frac{a}{\sqrt{\cos 2\varphi}} d\varphi.$$

注意到在 $-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$ 内恒有 $x - y \geq 0$, 于是, 所求的表面积为

$$\begin{aligned} P &= 2 \cdot 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x-y}{\sqrt{2}} \frac{a}{\sqrt{\cos 2\varphi}} d\varphi \\ &= \frac{4\pi a^2}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos \varphi - \sin \varphi) d\varphi \\ &= \frac{4\pi a^2}{\sqrt{2}} (\sin \varphi + \cos \varphi) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= 4\pi a^2. \end{aligned}$$

2499. 由抛物线 $ay = a^2 - x^2$ 及 Ox 轴所包围的图形绕 Ox 轴旋转而构成一旋转体. 求其表面积与等体积球的表面

积之比。

解 首先求此旋转体的表面积。

$$\sqrt{1+y'^2} = \frac{2\sqrt{x^2 + \frac{a^2}{4}}}{a},$$

从而

$$\begin{aligned} P_x &= 2 \cdot 2\pi \int_0^a \left(a - \frac{x^2}{a}\right) \cdot \frac{2\sqrt{x^2 + \frac{a^2}{4}}}{a} dx \\ &= 8\pi \int_0^a \sqrt{x^2 + \frac{a^2}{4}} dx - \frac{8\pi}{a^2} \int_0^a x^2 \sqrt{x^2 + \frac{a^2}{4}} dx \\ &= 8\pi \left\{ \frac{x}{2} \sqrt{x^2 + \frac{a^2}{4}} + \frac{a^2}{8} \ln \left(x + \sqrt{x^2 + \frac{a^2}{4}} \right) \right\} \Big|_0^a \\ &\quad - \frac{8\pi}{a^2} \left\{ \frac{x(2x^2 + \frac{a^2}{4})}{8} \sqrt{x^2 + \frac{a^2}{4}} \right. \\ &\quad \left. - \frac{a^4}{128} \ln \left(x + \sqrt{x^2 + \frac{a^2}{4}} \right) \right\} \Big|_0^a \\ &= \frac{\pi a^2}{8} \left[7\sqrt{5} + \frac{17}{2} \ln(2 + \sqrt{5}) \right]; \end{aligned}$$

其次，求旋转体的体积。

$$V_x = \pi \int_{-a}^a \left(a - \frac{x^2}{a}\right)^2 dx = \frac{16\pi a^3}{15}.$$

设与其等体积球的半径为 R ，则

$$\frac{4\pi R^3}{3} = \frac{16\pi a^3}{15}.$$

所以

$$R = \sqrt[3]{\frac{4}{3}}a.$$

于是, 此球的表面积为

$$P = 4\pi R^2 = 4\pi \sqrt[3]{\frac{16}{25}}a^2.$$

最后得到

$$\frac{P_x}{P} = \frac{\frac{\pi a^2}{8} \left[7\sqrt{5} + \frac{17}{2} \ln(2 + \sqrt{5}) \right]}{\frac{8\pi a^2}{5} \sqrt[3]{10}}$$

$$= \frac{5(14\sqrt{5} + 17 \ln(2 + \sqrt{5}))}{128 \cdot \sqrt[3]{10}}$$

$$\approx 1.013.$$

*) 利用1820题的结果.

2500. 由直线 $x = \frac{p}{2}$ 与抛物线 $y^2 = 2px$ 所包围的图形绕直线 $y = p$ 而旋转, 求这旋转体的体积和表面积.

$$\text{解 } V_{\text{旋转}} = \int_0^{\frac{p}{2}} \pi (p + \sqrt{2px})^2 dx$$

$$+ \int_0^{\frac{p}{2}} \pi (p - \sqrt{2px})^2 dx$$

$$= 4\pi p \int_0^{\frac{p}{2}} \sqrt{2px} dx$$

$$= \frac{4}{3} \pi p^3.$$

旋转体的侧面积为

$$\begin{aligned}
 S_{\text{侧}} &= \int_{(1)} 2\pi(p + \sqrt{2px}) ds \\
 &\quad + \int_{(1)} 2\pi(p - \sqrt{2px}) ds \\
 &= 4\pi p \int_{(1)} ds = 4\pi p \int_0^p \sqrt{1 + \frac{y^2}{p^2}} dy \\
 &= 4\pi \int_0^p \sqrt{y^2 + p^2} dy \\
 &= 4\pi \left(\frac{y}{2} \sqrt{y^2 + p^2} \right. \\
 &\quad \left. + \frac{p^2}{2} \ln(y + \sqrt{y^2 + p^2}) \right) \Big|_0^p \\
 &= 2\pi p^2 [\sqrt{2} + \ln(1 + \sqrt{2})],
 \end{aligned}$$

而底面积为

$$S_{\text{底}} = \pi(2p)^2 = 4\pi p^2,$$

于是, 所求的表面积为

$$\begin{aligned}
 P &= S_{\text{侧}} + S_{\text{底}} \\
 &= 2\pi p^2 [(2 + \sqrt{2}) + \ln(1 + \sqrt{2})].
 \end{aligned}$$

§9. 矩的计算法. 重心的坐标

1° 矩 若在 Oxy 平面上, 密度为 $\rho = \rho(y)$ 的质量 M 充满了某有界连续统 Ω (曲线, 平面的区域), 而 $\omega = \omega(y)$ 为 Ω 中纵标不超过 y 的部分的对应的度量 (弧长, 面积), 则数

$$M_k = \lim_{\max |\Delta y_i| \rightarrow 0} \sum_{i=1}^n \rho(y_i) y_i^k \Delta \omega(y_i)$$

$$= \int_a^b \rho y^k d\omega(y) \quad (k=0, 1, 2, \dots)$$

称为质量 M 对于 Ox 轴的 k 次矩。

特殊情形，当 $k=0$ 时得质量 M ，当 $k=1$ 时得静力矩，当 $k=2$ 时得转动惯量。

同样地可定义出质量对于坐标平面的矩。

若 $\rho=1$ ，则对应的矩称为几何矩（线矩，面积矩，体积矩等等）。

2° 重心 均匀平面图形 S 的重心的坐标 (x_0, y_0) 根据下面的公式来定义

$$x_0 = \frac{M_1^{(y)}}{S}, \quad y_0 = \frac{M_1^{(x)}}{S},$$

式中 $M_1^{(y)}$ ， $M_1^{(x)}$ 为面积 S 对于 Oy 轴和 Ox 轴的几何静力矩。

2501. 求半径为 a 的半圆弧对于过此弧两端点直径的静力矩和转动惯量。

解 取此直径所在的直线作为 Ox 轴，圆心作为原点，则圆的方程为

$$x^2 + y^2 = a^2.$$

从而

$$y = \sqrt{a^2 - x^2}$$

及

$$ds = \sqrt{1 + y'^2} dx = \frac{a}{y} dx = \frac{a}{\sqrt{a^2 - x^2}} dx.$$

于是，所求的静力矩和转动惯量*) 为

$$M_1 = \int_{-a}^a \sqrt{a^2 - x^2} \cdot \frac{a}{\sqrt{a^2 - x^2}} dx = 2a^2$$

及

$$\begin{aligned} M_2 &= \int_{-a}^a (a^2 - x^2) \cdot \frac{a}{\sqrt{a^2 - x^2}} dx \\ &= 2a \int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi a^3}{2}. \end{aligned}$$

2502. 求底为 b ，高为 h 的均匀三角形薄板对于其底边的静力矩和转动惯量 ($\rho = 1$).

解 取坐标系如图 4.46 所示.

$$\begin{aligned} M_1^{(x)} &= \frac{1}{2} \int_0^c y^2 dx \\ &= \frac{1}{2} \int_0^c y_1^2 dx \\ &\quad + \frac{1}{2} \int_c^b y_2^2 dx. \end{aligned}$$

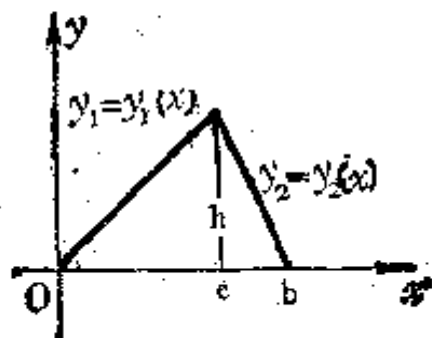


图 4.46

由于

$$y_1 = y_1(x) = \frac{h}{c} x,$$

*) 这里假定 $\rho = 1$ ，今后有类似情况，不再说明。

$$y_2 = y_2(x) = \frac{h}{c-b}(x-b),$$

于是, 所求的静力矩为

$$\begin{aligned} M_1^{(x)} &= \frac{1}{2} \int_0^b \frac{h^2}{c^2} x^2 dx \\ &\quad + \frac{1}{2} \int_0^b \frac{h^2}{(c-b)^2} (x-b)^2 dx \\ &= \frac{bh^2}{6}. \end{aligned}$$

又由于

$$x_1 = x_1(y) = \frac{c}{h}y,$$

$$x_2 = x_2(y) = b + \frac{c-b}{h}y,$$

于是, 所求的转动惯量为

$$\begin{aligned} M_2^{(x)} &= \int_0^h y^2 (x_2 - x_1) dy \\ &= \int_0^h y^2 \left(b - \frac{b}{h}y \right) dy = \frac{bh^2}{12}. \end{aligned}$$

2503. 求半轴长为 a 和 b 的均匀椭圆形薄板对于其主轴的转动惯量 ($\rho = 1$).

解 不妨设椭圆的方程为

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

则上、下半椭圆方程为

$$x_1 = -\frac{a}{b} \sqrt{b^2 - y^2},$$

$$x_2 = \frac{a}{b} \sqrt{b^2 - y^2}.$$

于是，所求的转动惯量为

$$\begin{aligned} M_2^{(x)} &= \int_{-b}^b y^2 (x_2 - x_1) dy \\ &= 2 \int_{-b}^b \frac{a}{b} y^2 \sqrt{b^2 - y^2} dy \\ &= 4ab^3 \int_0^{\frac{\pi}{2}} \sin^2 \varphi \cos^2 \varphi d\varphi^{**} \\ &= \frac{\pi a^3 b^3}{4}. \end{aligned}$$

至于 $M_2^{(y)}$ ，由对称性知：只须在 $M_2^{(x)}$ 的结果中将 a, b 对调即得。所以

$$M_2^{(y)} = \frac{\pi a^3 b}{4}.$$

*) 设 $y = b \sin \varphi$ 。

2504. 求底半径为 r 和高为 h 的均匀圆锥对于其底平面的静力矩和转动惯量 ($\rho = 1$)。

解 取坐标系如图 4.47 所示，则

$$M_1 = \int_0^h x \cdot P(x) dx,$$

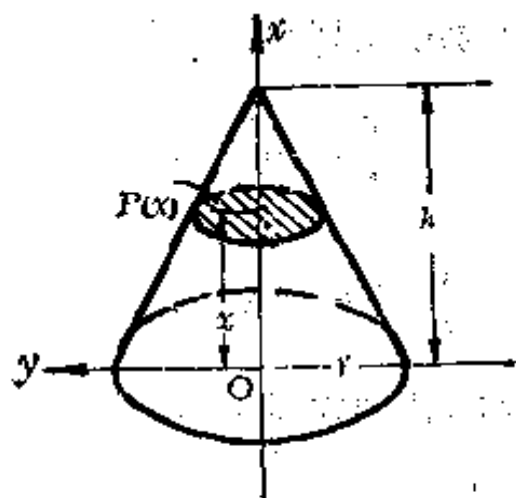


图 4.47

其中

$$P(x) = \pi y^2 = \pi \left[\frac{r}{h}(h-x) \right]^2.$$

于是, 所求的静力矩和转动惯量分别为

$$M_1 = \frac{\pi r^2}{h^2} \int_0^h x (h-x)^2 dx = \frac{\pi r^2 h^2}{12},$$

$$M_2 = \int_0^h x^2 \cdot P(x) dx.$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 (h-x)^2 dx = \frac{\pi r^2 h^3}{30}.$$

2505. 证明古尔金第一定理: 弧 C 绕着不与它相交的轴旋转而成的旋转面的面积, 等于这个弧的长度与这弧的重心所划出的圆周之长的乘积.

证 重心 (ξ, η) 具有这样的性质, 即如把曲线的全部“质量”都集中到它上面, 则此质量对于任何一个轴的静力矩, 都与曲线对此轴的静力矩相同. 即

$$\xi s = M_y = \int_0^l x ds,$$

$$\eta s = M_x = \int_0^l y ds,$$

式中 s 表示弧长。

于是

$$2\pi\eta \cdot s = 2\pi \int_0^l y ds.$$

上式的右端是弧 C 旋转而成的曲面面积，左端 $2\pi\eta$ 表示弧 C 绕 Ox 轴旋转时其重心所划出的圆周之长。从而定理得证。

2506. 证明古尔金第二定理：面积 S 绕不与它相交的轴旋转而成的旋转体，其体积等于面积 S 与这面积的重心所划出的圆周之长的相乘积。

证 由于

$$\eta \cdot S = M_x = \frac{1}{2} \int_a^b y^2 dx,$$

所以

$$2\pi\eta \cdot S = \pi \int_a^b y^2 dx.$$

上式右端即为旋转体的体积，从而定理得证。

2507. 求圆弧： $x = a \cos \varphi$, $y = a \sin \varphi$ ($|\varphi| \leq \alpha \leq \pi$) 重心的坐标。

解 显见

$$\eta = 0,$$

圆弧长

$$s = 2a\alpha.$$

由于

$$M_y = \int_0^s x ds = \int_{-\alpha}^{\alpha} a^2 \cos \varphi d\varphi = 2a^2 \sin \alpha,$$

所以

$$\xi = \frac{2a^2 \sin \alpha}{2a\alpha} = \frac{a \sin \alpha}{\alpha}.$$

即重心为 $(\frac{a \sin \alpha}{\alpha}, 0)$.

2508. 求抛物线: $ax = y^2$, $ay = x^2$ ($a > 0$) 所围成面积的重心的坐标.

解 利用古尔金第二定理来解此题. 首先, 此面积为

$$S = \frac{a^2}{3},$$

体积为

$$V = \pi \int_0^a \left(ax - \frac{x^4}{a^2} \right) dx = \frac{3\pi a^3}{10}.$$

于是

$$2\pi\eta \cdot \frac{a^2}{3} = \frac{3\pi a^3}{10},$$

所以

$$\eta = \frac{9a}{20}.$$

利用对称性知

$$\xi = \eta = \frac{9a}{20}.$$

即所求的重心为 $(\frac{9a}{20}, \frac{9a}{20})$.

*) 利用2397题的结果.

2509. 求面积

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad (0 \leq x \leq a, \quad 0 \leq y \leq b)$$

的重心的坐标.

解 首先, 我们已知第一象限椭圆的面积等于 $\frac{\pi ab}{4}$.

其次, 我们再求椭圆绕 Ox 轴旋转所得的旋转体体积. 因为

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

所以

$$V = \pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx = \frac{4}{3} \pi ab^2.$$

按古尔金第二定理, 我们有

$$2\pi \eta \frac{\pi ab}{4} = \frac{2}{3} \pi ab^2,$$

所以

$$\eta = \frac{4b}{3\pi}.$$

同理可求得

$$\xi = \frac{4a}{3\pi}.$$

事实上，只须在结果中将 a 和 b 对调即得。于是，所求的重心为 $(\frac{4a}{3\pi}, \frac{4b}{3\pi})$ 。

2510. 求半径为 a 的均匀半球的重心坐标。

解 取圆心作为原点，则球的方程为

$$x^2 + y^2 + z^2 = a^2.$$

设重心为 (ξ, η, ζ) ，显见 $\xi = \eta = 0$ 。而

$$V_{\text{半球}} = \frac{2\pi a^3}{3}.$$

将圆

$$y^2 + z^2 = a^2$$

绕 Oz 轴旋转，即得球。

又

$$\begin{aligned} M_1^{(z)} &= \int_{(V)} z dV = \pi \int_0^a z y^2 dz \\ &= \pi \int_0^a z(a^2 - z^2) dz = \frac{\pi a^4}{4}. \end{aligned}$$

最后得到

$$\zeta = \frac{M_1^{(z)}}{V} = \frac{\frac{\pi a^4}{4}}{\frac{2\pi a^3}{3}} = \frac{3a}{8}.$$

于是，所求重心为 $(0, 0, \frac{3a}{8})$ 。

2511. 求对数螺线

$$r = ae^{m\varphi} \quad (m > 0)$$

上由点 $O(-\infty, 0)$ 到点 $P(\varphi, r)$ 的弧 OP 的重心 $C(\varphi_0, r_0)$ 之坐标。当 P 点移动时， C 点画出怎样的曲线？

解 重心的直角坐标为

$$\begin{aligned} \xi &= \frac{\int_{(t)} x ds}{\int_{(t)} ds} \\ &= \frac{\int_{-\infty}^{\varphi} r \cos \varphi \cdot \sqrt{a^2(1+m^2)} e^{m\varphi} d\varphi}{\int_{-\infty}^{\varphi} \sqrt{a^2(1+m^2)} e^{m\varphi} d\varphi} \\ &= \frac{a \int_{-\infty}^{\varphi} e^{2m\varphi} \cos \varphi d\varphi}{\int_{-\infty}^{\varphi} e^{m\varphi} d\varphi} \\ &= \frac{mae^{m\varphi} (\sin \varphi + 2m \cos \varphi)}{4m^2 + 1} \end{aligned}$$

同法可得

$$\eta = \frac{\int_{(t)} y ds}{\int_{(t)} ds} = \frac{mae^{m\varphi} (2m \sin \varphi - \cos \varphi)}{4m^2 + 1}$$

于是，重心的极坐标为

$$r_0 = \sqrt{\xi^2 + \eta^2} = \frac{ma}{4m^2 + 1} \sqrt{4m^2 + 1} e^{m\varphi}$$

$$= \frac{mr}{\sqrt{4m^2 + 1}}$$

$$\operatorname{tg} \varphi_0 = \frac{\eta}{\xi} = \frac{2m \operatorname{tg} \varphi - 1}{\operatorname{tg} \varphi + 2m} = \frac{\operatorname{tg} \varphi - \frac{1}{2m}}{1 + \frac{1}{2m} \operatorname{tg} \varphi}$$

即 $\varphi_0 = \varphi - \alpha$, 其中 $\alpha = \operatorname{arc} \operatorname{tg} \frac{1}{2m}$.

当 P 点移动时, $C(\varphi_0, r_0)$ 画出的曲线为

$$r_0 = \frac{ma}{\sqrt{4m^2 + 1}} e^{m\varphi} = \frac{ma}{\sqrt{4m^2 + 1}} e^{m(\varphi_0 + \alpha)}$$

这也是一条对数螺线.

2512. 求曲线 $r = a(1 + \cos \varphi)$ 所围面积的重心坐标.

解 计算时, 将小扇形的重量集中在其重心

$(\frac{2}{3}r \cos \varphi, \frac{2}{3}r \sin \varphi)$ 处. 由对称性知 $\eta = 0$, 而

$$\begin{aligned} \xi &= \frac{\int_{(l)} xy dx}{\int_{(l)} y dx} \\ &= \frac{\frac{2}{3} \int_0^\pi r \cos \varphi \cdot \frac{1}{2} r^2 d\varphi}{\int_0^\pi \frac{1}{2} r^2 d\varphi} \\ &= \frac{2}{3} \frac{\int_0^\pi a^3 (1 + \cos \varphi)^2 \cos \varphi d\varphi}{\int_0^\pi a^2 (1 + \cos \varphi)^2 d\varphi} \end{aligned}$$

$$\begin{aligned}
&= \frac{2a}{3} \frac{\int_0^x (1 + 3\cos\varphi + 3\cos^2\varphi + \cos^3\varphi) \cos\varphi d\varphi}{\int_0^x (1 + 2\cos\varphi + \cos^2\varphi) d\varphi} \\
&= \frac{5a}{6}.
\end{aligned}$$

于是, 重心的极坐标为 $\varphi_0 = 0$, $r_0 = \frac{5a}{6}$.

2513. 求摆线 $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$) 的第一拱与 Ox 轴所围成面积的重心的坐标.

解 由对称性知 $\xi = \pi a$. 由于面积 $S = 3\pi a^2$ *) 及面积 S 绕 Ox 轴旋转而成的曲面包围的体积 $V_x = 5\pi^2 a^3$ **), 利用古尔金第二定理, 即得重心 (ξ, η) 适合下列关系式

$$2\pi\eta \cdot S = V_x$$

或

$$\eta = \frac{V_x}{2\pi S} = \frac{5\pi^2 a^3}{2\pi \cdot 3\pi a^2} = \frac{5a}{6}.$$

于是, 重心为 $(\pi a, \frac{5a}{6})$.

*) 利用2413题的结果.

**) 利用2480题(a)的结果.

***) 参看2506题.

2514. 求面积 $0 \leq x \leq a$; $y^2 \leq 2px$ 绕 Ox 轴旋转所成旋转体的重心的坐标.

解 由对称性知 $\eta = 0$. 又

$$\begin{aligned}\xi &= \frac{\int_0^a x \pi y^2 dx}{\int_0^a \pi y^2 dx} = \frac{\int_0^a 2px^2 dx}{\int_0^a 2px dx} \\ &= \frac{2}{3} a.\end{aligned}$$

于是, 所求的重心为 $(\frac{2}{3} a, 0)$.

2515. 求半球 $x^2 + y^2 + z^2 = a^2$ ($z \geq 0$) 的重心的坐标.

解 由对称性知

$$\xi = \eta = 0.$$

$$\begin{aligned}\xi &= \frac{\int_0^a 2 \cdot 2\pi x \sqrt{1+x'^2} dz}{\int_0^a 2\pi x \sqrt{1+x'^2} dz} \quad *) \\ &= \frac{\int_0^a 2\pi z \sqrt{a^2 - z^2} \cdot \frac{a}{\sqrt{a^2 - z^2}} dz}{\int_0^a 2\pi \sqrt{a^2 - z^2} \cdot \frac{a}{\sqrt{a^2 - z^2}} dz} \\ &= \frac{2\pi a \int_0^a z dz}{2\pi a \int_0^a dz} = \frac{2\pi a \cdot \frac{1}{2} a^2}{2\pi a^2} = \frac{a}{2}.\end{aligned}$$

于是, 所求的重心为 $(0, 0, \frac{a}{2})$.

*) 在此是将 $x^2 + z^2 = a^2$ 绕 Oz 轴旋转而得半球面.

§10. 力学和物理学中的问题

作成适当的积分和并找出它们的极限，来解下列问题：

2516. 轴的长度 $l=10$ 米，若该轴的线性密度按定律 $\delta=6+0.3x$ 千克/米而变更，其中 x 为距轴两端点中之一端的距离，求轴的质量。

解 将轴 n 等分，每份的长 $\Delta x = \frac{10}{n}$ 。把每小段近似地看成是均匀的，并以右端点的密度作为小段的密度。这样，便得到轴的质量 M 的近似值，即

$$M \approx \sum_{i=1}^n \left(6 + 0.3 \times \frac{10}{n} i \right) \frac{10}{n}.$$

显然， n 愈大愈近似，于是，得轴的质量

$$\begin{aligned} M &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 + 0.3 \times \frac{10}{n} i \right) \frac{10}{n} \\ &= \lim_{n \rightarrow \infty} \left[60 + \frac{15 \times (n+1)}{n} \right] = 75 \text{ (千克)}. \end{aligned}$$

2517. 把质量为 m 的物体从地球（其半径为 R ）表面升高到高度为 h 的地位，需要化费多大的功？若物体远离至无穷远去，则功等于什么？

解 由牛顿万有引力定律

$$f = k \frac{mM}{r^2},$$

其中 M 为地球的质量， r 为物体离开地球中心的距

离， k 为比例常数。将 h 分成 n 等份，在每份上把引力近似地看作是不变的，在第 i 份上取

$$r_i = \sqrt{\left[\frac{h}{n}(i-1) + R \right] \left[\frac{h}{n}i + R \right]}, \text{ 则力}$$

$$f_i = k \frac{mM}{\left[\frac{h}{n}(i-1) + R \right] \cdot \left[\frac{h}{n}i + R \right]}$$

于是所要求的功

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n k \frac{mM}{\left[\frac{h}{n}(i-1) + R \right] \left[\frac{hi}{n} + R \right]} \cdot \frac{h}{n} \\ &= \lim_{n \rightarrow \infty} k m M n \sum_{i=1}^n \left[\frac{1}{h(i-1) + nR} \right. \\ &\quad \left. - \frac{1}{hi + nR} \right] \\ &= \lim_{n \rightarrow \infty} k m M n \left[\frac{1}{nR} - \frac{1}{n(R+h)} \right] \\ &= \frac{kmMh}{(R+h)R} \end{aligned}$$

其中 g 为重力加速度， $k = \frac{gR^2}{M}$ 为引力常数。若物体远离至无穷远去，则功

$$A_{\infty} = \lim_{h \rightarrow \infty} W = \lim_{h \rightarrow \infty} \frac{kmMh}{(R+h)R} = mgR.$$

2518. 若 1 千克的力能使弹簧伸长 1 厘米，现在要使这弹簧伸长 10 厘米，问需要花费多大的功？

解 由虎克定律知, 弹性恢复力 F 与伸长量 x 成正比, 即

$$F = kx.$$

由条件知: $k = 1$, 因而 $F = x$.

现将10厘米 n 等分, 每份上恢复力的大小近似地看作是不变的, 并取右端点来作和, 即得功 W 的近似值

$$W \approx \sum_{i=1}^n \frac{10}{n} i \cdot \frac{10}{n}.$$

显然, n 愈大愈近似. 于是, 所要求的功

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{10}{n} i \cdot \frac{10}{n} = \lim_{n \rightarrow \infty} 50 \frac{n+1}{n}$$

$$= 50 \text{ (千克厘米)} = 0.5 \text{ (千克米)}.$$

2519. 直径为 20 厘米, 长为 80 厘米的圆柱被压力为 10 千克/厘米² 的蒸汽充满着. 假定气体的温度不变, 要使气体的体积减小一半, 须要化费多大的功?

解 由波义耳—马利奥特定律有

$$pv = C,$$

其中 p 表示气体的压力, v 表示体积, C 为常量. 由条件知, 常量

$$C = 10 \cdot \pi \cdot 100 \cdot 80 = 800\pi \text{ (千克米)}.$$

设初始时气体体积为 v_0 , 将区间 $\left[\frac{v_0}{2}, v_0\right]$ 分成 n 个小区间, 分点依次为

$$\frac{v_0}{2}, \frac{v_0}{2}q, \frac{v_0}{2}q^2, \dots, \frac{v_0}{2}q^{i-1}, \dots, \frac{v_0}{2}q^n = v_0,$$

其中 $q = \sqrt[n]{\frac{v_0}{\frac{v_0}{2}}} = \sqrt[n]{2}$ 。由于气体体积从 $\frac{v_0}{2}q^{i+1}$ 减小

至 $\frac{v_0}{2}q^i$ 须要化费功的近似值为

$$C\left(\frac{v_0}{2}q^i\right)^{-1}\left(\frac{v_0}{2}q^{i+1} - \frac{v_0}{2}q^i\right),$$

于是，所要求的功

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} C \left(\frac{v_0}{2}q^i\right)^{-1} \left(\frac{v_0}{2}q^{i+1} - \frac{v_0}{2}q^i\right) \\ &= \lim_{n \rightarrow \infty} Cn \left(\sqrt[n]{2} - 1\right) = C \ln 2 \quad *) \\ &= 800 \pi \cdot \ln 2 \approx 1742 \text{ (千克米)}. \end{aligned}$$

*) 利用541题的结果。

2520. 求水对于垂直壁上的压力，这壁的形状为半圆形，半径为 a 且其直径位于水的表面上。

解 为求出水对半圆形的压力，只要计算出作用于四分之一圆上的压力，然后再把它两倍起来。现将四分之一圆等分成 n 个小圆心角为 $\Delta\theta$ 的小扇形（图4.48），作用于该小扇形上的压力的近似值为

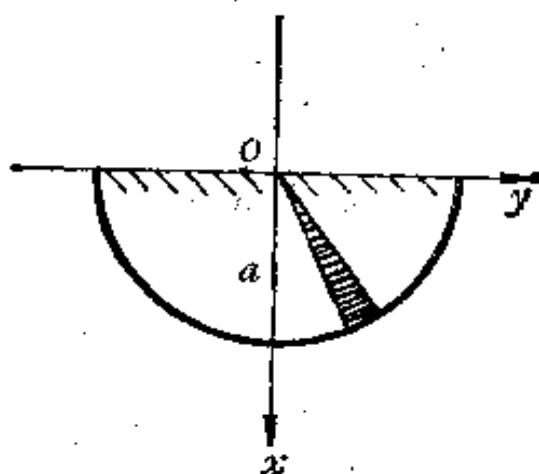


图 4.48

$$\frac{1}{2}a^2 \Delta\theta \cdot \frac{2}{3}a \sin \theta_i,$$

其中 $\Delta\theta = \frac{\pi}{2n}$, $\theta_i = \frac{i\pi}{2n}$.

于是, 作用于半圆上的压力

$$P = 2 \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2}a^2 \cdot \frac{2}{3}a \sin \frac{i\pi}{2n} \cdot \frac{\pi}{2n}$$

$$= \frac{2a^3}{3} \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{i\pi}{2n} \cdot \frac{\pi}{2n} = \frac{2a^3}{3} \quad *)$$

*) 利用2187题的结果.

2521. 求水对于垂直壁上的压力, 这壁的形状为梯形, 其下底 $a = 10$ 米, 上底 $b = 6$ 米, 高 $h = 5$ 米, 下底沉没于水面下的距离为 $c = 20$ 米.

解 取坐标系如图 4.49 所示. AB 所满足的方程为

$$y = \frac{4}{5}x - 6.$$

将区间 $[15, 20]$ n 等分, 每份长

$\Delta x = \frac{5}{n}$, 对应于 Δx

的小条上所受的压力的近似值为

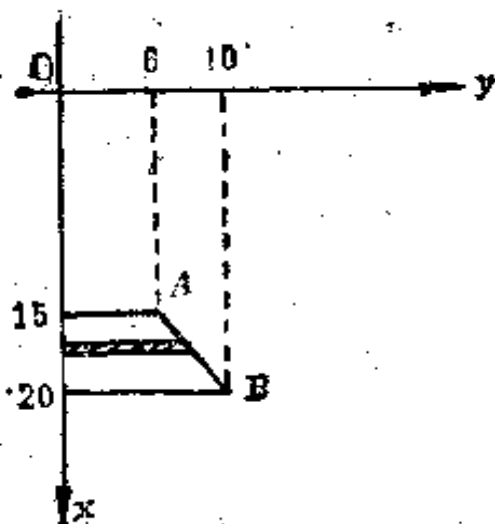


图 4.49

$$\left[\frac{4}{5} \left(15 + \frac{5i}{n} \right) - 6 \right] \left(15 + \frac{5i}{n} \right) \frac{5}{n}$$

于是，所要求的压力

$$\begin{aligned} P &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\frac{4}{5} \left(15 + \frac{5i}{n} \right) - 6 \right] \left(15 + \frac{5i}{n} \right) \frac{5}{n} \\ &= 708 \frac{1}{3} \text{ (吨)}^* \end{aligned}$$

*) 仿照2185题和2518题的作法。

作出微分方程式以解下列问题：

2522. 点运动的速度是按下面的规律而变化：

$$v = v_0 + at.$$

问在闭间隔 $[0, T]$ 内这点经过的路程怎样？

解 设路程为 s ，则由导数的力学意义知

$$\frac{ds}{dt} = v = v_0 + at,$$

即 dt 时间内经历的路程

$$ds = (v_0 + at) dt,$$

于是，

$$\begin{aligned} s &= \int_0^T (v_0 + at) dt \\ &= v_0 T + \frac{1}{2} a T^2. \end{aligned}$$

2523. 半径为 R 而密度为 δ 的均匀球体以角速度 ω 绕其直径而旋转。求此球的动能。

解 已知半径为 R 质量为 M 的盘绕垂直盘心的轴的转动惯量为 $\frac{1}{2}MR^2$ 。不妨设球面方程为 $x^2 + y^2 + z^2 = R^2$ ，则考察以 dz 为厚度的垂直于 z 轴的圆盘，其转动惯量为

$$\begin{aligned} dJ_z &= \frac{1}{2}\pi(R^2 - z^2)\delta \cdot (R^2 - z^2)dz \\ &= \frac{1}{2}\pi\delta(R^2 - z^2)^2 dz. \end{aligned}$$

从而球体的转动惯量

$$J_z = \int_{-R}^R \frac{1}{2}\pi\delta(R^2 - z^2)^2 dz = \frac{8}{15}\pi\delta R^5.$$

于是，球的动能

$$E = \frac{1}{2}J\omega^2 = \frac{4}{15}\pi\delta\omega^2 R^5.$$

注 原题误为球壳，现根据原答案予以改正。

2524. 具不变的线性密度 μ_0 的无穷直线以怎样的力吸引距此直线距离为 a 质量为 m 的质点？

解 取坐标系如图 4.50 所示， $|AO| = a$ 。设引力在坐

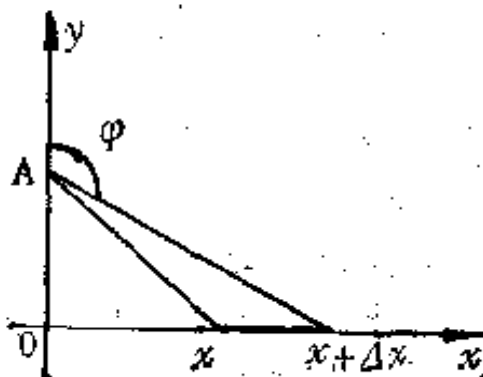


图 4.50

标轴上的射影为 F_x 、 F_y 。由于

$$\begin{aligned} dF_x &= k \frac{m\mu_0 dx}{(a^2+x^2)} \cos \varphi \\ &= -\frac{k m \mu_0 a}{(a^2+x^2)^{\frac{3}{2}}} dx, \end{aligned}$$

于是,

$$\begin{aligned} F_x &= -2km\mu_0 a \int_0^{+\infty} \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} \\ &= -2km\mu_0 a \cdot \frac{x}{a^2 \sqrt{a^2+x^2}} \Big|_0^{+\infty} \\ &= -\frac{2km\mu_0}{a}. \end{aligned}$$

由对称性知, $F_x=0$ 。事实上, 我们有

$$\begin{aligned} F_x &= \int_{-\infty}^{+\infty} \frac{k m \mu_0 \sin \varphi}{a^2+x^2} dx \\ &= km\mu_0 \int_{-\infty}^{+\infty} \frac{x}{(a^2+x^2)^{\frac{3}{2}}} dx = 0. \end{aligned}$$

其中 k 为引力常数。由上述分析知, 引力指向 y 轴的负向。

2525. 计算半径为 a 及固定的表面密度为 δ_0 的圆形薄板以怎样的力吸引质量为 m 的质点 P , 此质点位于通过薄板中心 Q 且垂直于薄板平面的垂直线上, 最短距离 PQ 等于 b 。

解 取坐标系如图4.51所示, 显然, 引力指向 y 轴的正向。对于以 x 为半径的圆环, 其质量为 $dm = \delta_0 \cdot 2\pi x dx$, 对质点 P 的引力

$$\begin{aligned} dF_y &= 2km\delta_0 \pi \frac{\cos\theta}{b^2+x^2} dx \\ &= 2km\delta_0 \pi \frac{bx}{(b^2+x^2)^{\frac{3}{2}}} dx, \end{aligned}$$

于是, 所要求的引力

$$\begin{aligned} F_y &= 2km\delta_0 \pi \int_0^a \frac{bx}{(b^2+x^2)^{\frac{3}{2}}} dx \\ &= 2km\delta_0 \pi \left(1 - \frac{b}{\sqrt{a^2+b^2}} \right). \end{aligned}$$

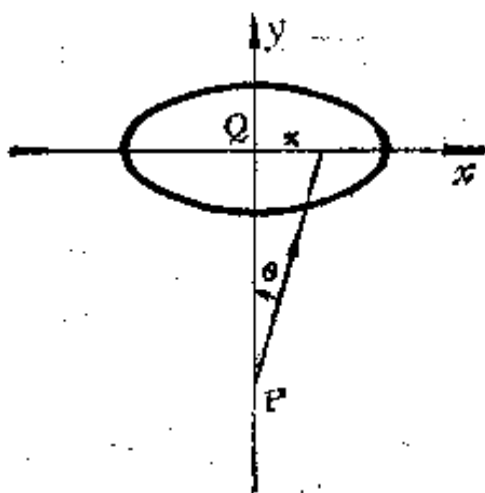


图 4.51

2526. 根据托里拆利定律, 液体从容器中流出的速度等于

$$v = c\sqrt{2gh},$$

式中 g 为重力加速度, h 为液体表面在开孔上之高,

$c=0.6$ 为实验系数。

直径为 $D=1$ 米及高为 $H=2$ 米的直立圆柱形大桶，充满之后从其底上直径为 $d=1$ 厘米的圆孔流出，须要多长时间，完全流空？

解 取坐标系如图4.52

所示。对于 dt 时间，从圆孔流出的液体体积

$$dv = 0.15\pi\sqrt{2gx} dt,$$

而桶内液体体积的减少量为

$$dv = -\pi(50)^2 dx,$$

其中 x 随时间 t 的增大而减小。流出的量应等于桶内减少的量，于是

$$\begin{aligned} -0.15\pi\sqrt{2gx} dt \\ = \pi(50)^2 dx. \end{aligned}$$

积分，得

$$\int_0^t dt = -\int_{200}^x \frac{2500}{0.15} \frac{dx}{\sqrt{2gx}},$$

即

$$t = -33333 \frac{1}{\sqrt{2g}} (\sqrt{x} - \sqrt{200}),$$

其中 $g=980$ 厘米/秒²。当 $x=0$ 时， t 表示水流完所需的时间。因而所要求的时间

$$t = \frac{33333\sqrt{200}}{\sqrt{2 \times 980}} = 10648 \text{ (秒)}.$$

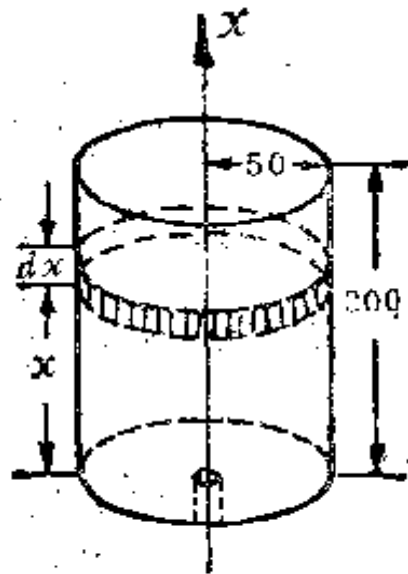


图 4.52

2527. 旋转体的容器应当是什么形状, 才能使液体流出时, 液体表面的下降是均匀的?

解 取坐标系如图4.53所示. 不妨设流出孔的半径为单位厘米.

仿上题分析, 得

$$\begin{aligned} \pi x^2 dy &= -\pi v dt \\ &= -\pi c \sqrt{2gy} dt, \end{aligned}$$

即

$$dy = -c \sqrt{\frac{y}{2g}}$$

$$\cdot \frac{\sqrt{y}}{x^2} dt.$$

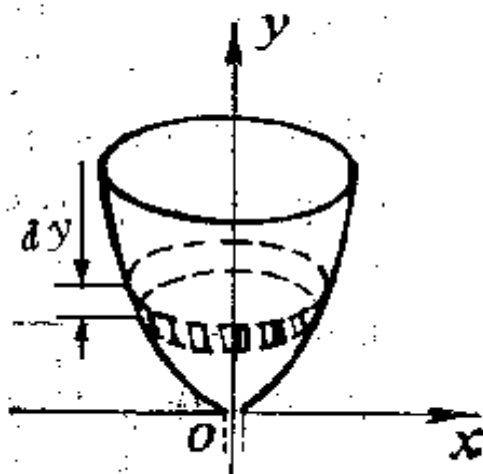


图 4.53

其中 c 为实验系数, g 为重力加速度.

由题意知

$$\frac{dy}{dt} = -c \sqrt{2g} \frac{\sqrt{y}}{x^2}$$

应等于常数 k , 即

$$-c \sqrt{2g} \frac{\sqrt{y}}{x^2} = k,$$

于是

$$y = Cx^4,$$

其中 C 为常数. 所以, 容器应当是把曲线 $y = Cx^4$ 绕铅直轴 Oy 旋转而得的曲面所构成的.

2528. 镭在每一时刻的分解速度与其现存的数量成比例, 设在开始的时刻 $t = 0$ 有镭 Q_0 克, 经过时间 $T = 1600$ 年

它的量减少了一半。求镭分解的规律。

解 设 Q 为镭现存的量，按题设有

$$\frac{dQ}{dt} = -kQ,$$

其中 k 为比例系数。即

$$\frac{dQ}{Q} = -k dt,$$

两端积分

$$\int_{Q_0}^{\frac{Q_0}{2}} \frac{dQ}{Q} = \int_0^{1600} -k dt,$$

从而

$$k = -\frac{\ln 2}{1600}.$$

于是

$$\int_{Q_0}^Q \frac{dQ}{Q} = -\frac{\ln 2}{1600} \int_0^t dt,$$

$$\ln \frac{Q}{Q_0} = \ln 2 \cdot \frac{t}{1600},$$

所以，镭的分解规律为

$$Q = Q_0 \cdot 2^{\frac{t}{1600}}.$$

2529⁺. 变换物质 A 为物质 B 的二阶化学反应之速度与此二物质的浓度相乘之积成正比。问经过 $t = 1$ 小时在容器中所含有的物质 B 之百分率如何？设 $t = 0$ 分时有 20% 的物质 B ，而当 $t = 15$ 分它变成 80%。

解 设 x 为生成物 B 的浓度，按题设有

$$\frac{dx}{dt} = kx(1-x),$$

其中 k 为比例常数。即

$$\frac{dx}{x(1-x)} = k dt.$$

两端积分

$$\int_{0.2}^{0.8} \frac{dx}{x(1-x)} = \int_0^{15} k dt,$$

从而

$$k = \frac{1}{15} \ln 16.$$

于是,

$$\int_{0.2}^x \frac{dx}{x(1-x)} = \int_0^t k dt = \frac{t}{15} \ln 16,$$

即

$$t = \frac{15}{\ln 16} \ln \frac{4x}{1-x}.$$

以 $t = 60$ 代入上式, 得

$$x = \frac{16^4}{16^4 + 4} = 99.99\%.$$

所以, 经过 $t = 1$ 小时在容器中所含有的物质 B 之百分率为 99.99%.

2530. 根据虎克定律, 棒的相对伸长率 e 与在对应的横断面上的应力 σ 成比例, 即是说

$$\varepsilon = \frac{\sigma}{E},$$

式中 E 为杨氏模数。

求圆锥形重棒的伸长，此锥形的顶向下而底固定，设底半径等于 R ，圆锥的高为 H ，比重为 γ 。

解 取坐标系如图4.54所示。

设 $z=h$ 截面处，对于高度为 dh 的锥体伸长为 dl ，则有

$$\begin{aligned} \varepsilon &= \frac{dl}{dh} \\ &= \frac{\frac{1}{3}\pi r^2(H-h)\gamma}{\pi r^2 E} \\ &= \frac{1}{3} \frac{(H-h)}{E} \gamma, \end{aligned}$$

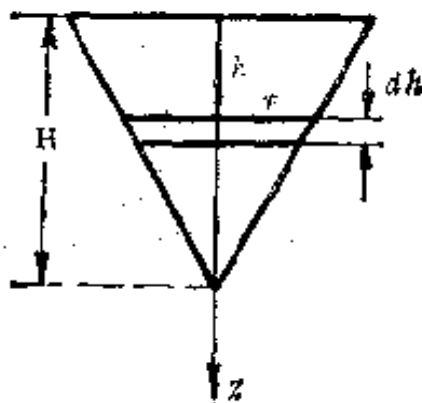


图 4.54

即

$$dl = \frac{1}{3} \frac{(H-h)}{E} \gamma dh.$$

于是圆锥形重棒总的伸长量为

$$l = \int_0^H \frac{1}{3} \frac{(H-h)\gamma}{E} dh = \frac{\gamma H^2}{6E}.$$

§11. 定积分的近似算法

1° 矩形公式 若函数 $y = y(x)$ 于有穷的闭区间 $[a, b]$

上连续且可微分充分多次数，并且 $h = \frac{b-a}{n}$ ， $x_i = a + ih$
 ($i = 0, 1, \dots, n$)， $y_i = y(x_i)$ ，则

$$\int_a^b y(x) dx = h(y_0 + y_1 + \dots + y_{n-1}) + R_n,$$

式中

$$R_n = -\frac{(b-a)^2}{2n} y'(\xi) \quad (a \leq \xi \leq b).$$

2° 梯形公式 用相同的记号有

$$\int_a^b y(x) dx = h\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1}\right) + R_n,$$

式中

$$R_n = -\frac{(b-a)^3}{12n^2} f''(\xi') \quad (a \leq \xi' \leq b).$$

3° 抛物线公式 (辛普森公式) 命 $n = 2k$ ，得

$$\int_a^b y(x) dx = \frac{h}{3} [(y_0 + y_{2k}) + 4(y_1 + y_3 + \dots + y_{2k-1}) + 2(y_2 + y_4 + \dots + y_{2k-2})] + R_n,$$

式中

$$R_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(\xi'') \quad (a \leq \xi'' \leq b).$$

2531. 利用矩形公式 ($n = 12$)，近似地计算

$$\int_0^{2\pi} x \sin x \, dx$$

并把结果同精确答数比较。

解 $h = \frac{\pi}{6}$.

$$x_0 = 0, \quad y_0 = 0;$$

$$x_1 = \frac{\pi}{6}, \quad y_1 = \frac{\pi}{6} \sin \frac{\pi}{6} = 0.2618;$$

$$x_2 = \frac{\pi}{3}, \quad y_2 = \frac{\pi}{3} \sin \frac{\pi}{3} = 0.9069;$$

$$x_3 = \frac{\pi}{2}, \quad y_3 = \frac{\pi}{2} \sin \frac{\pi}{2} = 1.5708;$$

$$x_4 = \frac{2\pi}{3}, \quad y_4 = \frac{2\pi}{3} \sin \frac{2\pi}{3} = 1.8138;$$

$$x_5 = \frac{5\pi}{6}, \quad y_5 = \frac{5\pi}{6} \sin \frac{5\pi}{6} = 1.3090;$$

$$x_6 = \pi, \quad y_6 = \pi \sin \pi = 0;$$

$$x_7 = \frac{7\pi}{6}, \quad y_7 = \frac{7\pi}{6} \sin \frac{7\pi}{6} = -1.8326;$$

$$x_8 = \frac{4\pi}{3}, \quad y_8 = \frac{4\pi}{3} \sin \frac{4\pi}{3} = -3.6276;$$

$$x_9 = \frac{3\pi}{2}, \quad y_9 = \frac{3\pi}{2} \sin \frac{3\pi}{2} = -4.7124;$$

$$x_{10} = \frac{5\pi}{3}, \quad y_{10} = \frac{5\pi}{3} \sin \frac{5\pi}{3} = -4.5345;$$

$$x_{11} = \frac{11\pi}{6}, \quad y_{11} = \frac{11\pi}{6} \sin \frac{11\pi}{6} = -2.8798.$$

按矩形公式，得

$$\begin{aligned} & \int_0^{2\pi} x \sin x \, dx \\ &= \frac{\pi}{6} (y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \\ & \quad + y_8 + y_9 + y_{10} + y_{11}) \\ &= -6.1390. \end{aligned}$$

实际上，

$$\begin{aligned} & \int_0^{2\pi} x \sin x \, dx \\ &= -x \cos x \Big|_0^{2\pi} + \int_0^{2\pi} \cos x \, dx \\ &= -6.2832. \end{aligned}$$

利用梯形公式计算下列积分并估计它们的误差：

2532. $\int_0^1 \frac{dx}{1+x} \quad (n=8).$

解 $h = \frac{1}{8} = 0.125.$

$$x_0 = 0, \quad y_0 = 1; \quad \frac{y_0 + y_8}{2} = 0.75,$$

$$x_8 = 1, \quad y_8 = 0.5;$$

$$x_1 = \frac{1}{8} = 0.125, \quad y_1 = 0.88889;$$

$$\begin{aligned}
x_2 &= 0.25, & y_2 &= 0.8; \\
x_3 &= 0.375, & y_3 &= 0.72727; \\
x_4 &= 0.5, & y_4 &= 0.66667; \\
x_5 &= 0.625, & y_5 &= 0.61538; \\
x_6 &= 0.75, & y_6 &= 0.57143; \\
x_7 &= 0.875, & y_7 &= 0.53333 \quad (+
\end{aligned}$$

$$\sum_{i=1}^7 y_i = 4.80297.$$

按梯形公式，得

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x} &= h \left(\frac{y_0 + y_8}{2} + \sum_{i=1}^7 y_i \right) \\
&= 0.125(0.75 + 4.80297) \\
&\approx 0.69412,
\end{aligned}$$

误差为

$$|R_n| = \left| \frac{1}{12 \times 8^2} \cdot \frac{2}{(1+\xi)^3} \right| \quad (0 \leq \xi \leq 1).$$

于是，

$$|R_n| \leq \frac{2}{12 \times 8^2} \leq 0.0027 = 2.7 \times 10^{-3}$$

实际上，

$$\int_0^1 \frac{dx}{1+x} = \ln(1+x) \Big|_0^1 = \ln 2 \approx 0.69315.$$

2533. $\int_0^1 \frac{dx}{1+x^3} \quad (n=12).$

解 $h = \frac{1}{12} = 0.08333.$

$$x_0 = 0, \quad y_0 = 1;$$

$$\frac{y_0 + y_{12}}{2} = 0.75,$$

$$x_{12} = 1, \quad y_{12} = \frac{1}{2} = 0.5;$$

$$x_1 = \frac{1}{12}, \quad y_1 = 0.99942;$$

$$x_2 = \frac{1}{6}, \quad y_2 = 0.99539;$$

$$x_3 = \frac{1}{4}, \quad y_3 = 0.98462;$$

$$x_4 = \frac{1}{3}, \quad y_4 = 0.96429;$$

$$x_5 = \frac{5}{12}, \quad y_5 = 0.93254;$$

$$x_6 = \frac{1}{2}, \quad y_6 = 0.88889;$$

$$x_7 = \frac{7}{12}, \quad y_7 = 0.83438;$$

$$x_8 = \frac{2}{3}, \quad y_8 = 0.77143;$$

$$x_9 = \frac{3}{4}, \quad y_9 = 0.70330;$$

$$x_{10} = \frac{5}{6}, \quad y_{10} = 0.63343;$$

$$x_{11} = \frac{11}{12}, \quad y_{11} = 0.56489 \quad (+$$

$$\sum_{i=1}^{11} y_i = 9.27258.$$

按梯形公式，得

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^3} &= h \left(\frac{y_0 + y_{12}}{2} + \sum_{i=1}^{11} y_i \right) \\ &= 0.0833(0.75 + 9.27258) \\ &\doteq 0.83518, \end{aligned}$$

误差为

$$|R_n| = \left| \frac{1}{12 \times 12^2} \cdot \frac{12\xi^4 - 6\xi}{(1+\xi^3)^3} \right| \quad (0 \leq \xi \leq 1).$$

利用求极值的方法，估计得 $\left| \frac{12\xi^4 - 6\xi}{(1+\xi^3)^3} \right|$ 在 $[0, 1]$ 上不超过 2。于是，

$$|R_n| \leq \frac{2}{12 \times 12^2} < 0.00116 = 1.16 \times 10^{-3}.$$

实际上，

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^3} &= \left[\frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} \right. \\ &\quad \left. + \frac{1}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{2x-1}{\sqrt{3}} \right] \Big|_0^1 \\ &= \frac{1}{3} \ln 2 + \frac{\pi}{3\sqrt{3}} \\ &\doteq 0.83565. \end{aligned}$$

* 利用1881题的结果。

$$2534. \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{4} \sin^2 x} dx \quad (n = 6).$$

解 $h = \frac{\pi}{12} = 0.2618,$

$$x_0 = 0, \quad y_0 = 1;$$

$$\frac{y_0 + y_6}{2} = 0.9330,$$

$$x_6 = \frac{\pi}{2}, \quad y_6 = 0.8660;$$

$$x_1 = \frac{\pi}{12}, \quad y_1 = 0.9916;$$

$$x_2 = \frac{\pi}{6}, \quad y_2 = 0.9682;$$

$$x_3 = \frac{\pi}{4}, \quad y_3 = 0.9354;$$

$$x_4 = \frac{\pi}{3}, \quad y_4 = 0.9014;$$

$$x_5 = \frac{5\pi}{12}, \quad y_5 = 0.8756 \quad (+$$

$$\sum_{i=1}^5 y_i = 4.6722.$$

按梯形公式，得

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{4} \sin^2 x} dx = h \left(\frac{y_0 + y_6}{2} + \sum_{i=1}^5 y_i \right)$$

$$= 0.2618(0.9330 + 4.6724)$$

$$\approx 1.4674.$$

误差为

$$|R_n| = \frac{\left(\frac{\pi}{2}\right)^3}{12 \times 6^2} |y''(\xi)|,$$

式中 $y = \sqrt{1 - \frac{1}{4} \sin^2 x}$, $0 \leq \xi \leq \frac{\pi}{2}$. 利用 $\frac{\sqrt{3}}{2} \leq y \leq 1$ 及 $y^2 = 1 - \frac{1}{4} \sin^2 x$, 依次求导可得 $|y''| \leq \frac{\sqrt{3}}{6}$. 于是,

$$|R_n| \leq \frac{\pi^3}{8 \times 12 \times 6^2} \cdot \frac{\sqrt{3}}{6} < 2.59 \times 10^{-8}.$$

利用辛普森公式计算下列积分:

2535. $\int_1^9 \sqrt{x} dx$ ($n=4$).

解 $h=2$.

$$x_0 = 1, y_0 = 1;$$

$$x_1 = 3, y_1 = \sqrt{3} = 1.732;$$

$$x_2 = 5, y_2 = \sqrt{5} = 2.236;$$

$$x_3 = 7, y_3 = \sqrt{7} = 2.646;$$

$$x_4 = 9, y_4 = 3.$$

按辛普森公式, 得

$$\begin{aligned} \int_1^9 \sqrt{x} dx &= \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2] \\ &= \frac{2}{3} [4 + 4(1.732 + 2.646) + 2(2.236)] \end{aligned}$$

$$\approx 17.323,$$

实际上,

$$\int_1^9 \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_1^9 = \frac{52}{3} \approx 17.333.$$

2536. $\int_0^{\pi} \sqrt{3 + \cos x} dx \quad (n=6).$

解 $h = \frac{\pi}{6}.$

$$x_0 = 0, \quad y_0 = 2;$$

$$x_1 = \frac{\pi}{6}, \quad y_1 = \sqrt{3 + \cos \frac{\pi}{6}} = \sqrt{3.866} = 1.966;$$

$$x_2 = \frac{\pi}{3}, \quad y_2 = \sqrt{3 + \cos \frac{\pi}{3}} = \sqrt{3.5} = 1.871;$$

$$x_3 = \frac{\pi}{2}, \quad y_3 = \sqrt{3 + \cos \frac{\pi}{2}} = \sqrt{3} = 1.732;$$

$$x_4 = \frac{2\pi}{3}, \quad y_4 = \sqrt{3 + \cos \frac{2\pi}{3}} = \sqrt{2.5} \\ = 1.581;$$

$$x_5 = \frac{5\pi}{6}, \quad y_5 = \sqrt{3 + \cos \frac{5\pi}{6}} = \sqrt{2.134} \\ = 1.461;$$

$$x_6 = \pi, \quad y_6 = \sqrt{3 + \cos \pi} = \sqrt{2} = 1.414.$$

按辛普森公式, 得

$$\int_0^{\pi} \sqrt{3 + \cos x} dx$$

$$\begin{aligned}
&= \frac{\pi}{18} [(2+1.414) + 4(1.966+1.736+1.461) + \\
&\quad + 2(1.871+1.581)] \\
&= 5.4053.
\end{aligned}$$

$$2537^+. \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx \quad (n=10).$$

解 $h = \frac{\pi}{20}.$

$$x_0 = 0, \quad y_0 = 1,$$

$$x_1 = \frac{\pi}{20}, \quad y_1 = \frac{20}{\pi} \sin \frac{\pi}{20} = 0.99589,$$

$$x_2 = \frac{\pi}{10}, \quad y_2 = \frac{10}{\pi} \sin \frac{\pi}{10} = 0.98363,$$

$$x_3 = \frac{3\pi}{20}, \quad y_3 = \frac{20}{3\pi} \sin \frac{3\pi}{20} = 0.96340,$$

$$x_4 = \frac{\pi}{5}, \quad y_4 = \frac{5}{\pi} \sin \frac{\pi}{5} = 0.93549,$$

$$x_5 = \frac{\pi}{4}, \quad y_5 = \frac{4}{\pi} \sin \frac{\pi}{4} = 0.90032,$$

$$x_6 = \frac{3\pi}{10}, \quad y_6 = \frac{10}{3\pi} \sin \frac{3\pi}{10} = 0.85839,$$

$$x_7 = \frac{7\pi}{20}, \quad y_7 = \frac{20}{7\pi} \sin \frac{7\pi}{20} = 0.81033,$$

$$x_8 = \frac{2\pi}{5}, \quad y_8 = \frac{5}{2\pi} \sin \frac{2\pi}{5} = 0.75683,$$

$$x_9 = \frac{9\pi}{20}, \quad y_9 = \frac{20}{9\pi} \sin \frac{9\pi}{20} = 0.69865,$$

$$x_{10} = \frac{\pi}{2}, \quad y_{10} = \frac{2}{\pi} = 0.63662.$$

按辛普森公式，得

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx &= \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 \\ &\quad + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)] \\ &= \frac{\pi}{60} [(1 + 0.63662) + 4(0.99589 \\ &\quad + 0.96340 + 0.90032 + 0.81033 \\ &\quad + 0.69865) + 2(0.98363 + 0.93549 \\ &\quad + 0.85839 + 0.75683)] \\ &\approx 1.37076. \end{aligned}$$

$$2538^+. \int_0^1 \frac{x dx}{\ln(1+x)} \quad (n=5).$$

解 $h = \frac{1}{6}.$

$$x_0 = 0, \quad y_0 = \lim_{x \rightarrow 0} \frac{x}{\ln(1+x)} = 1;$$

$$x_1 = \frac{1}{6}, \quad y_1 = 1.0812;$$

$$x_2 = \frac{1}{3}, \quad y_2 = 1.1587;$$

$$x_3 = \frac{1}{2}, \quad y_3 = 1.2332;$$

$$x_4 = \frac{2}{3}, \quad y_4 = 1.3051;$$

$$x_5 = \frac{5}{6}, \quad y_5 = 1.3748;$$

$$x_6 = 1, \quad y_6 = 1.4427.$$

按辛普森公式，得

$$\begin{aligned} \int_0^1 \frac{x dx}{\ln(1+x)} &= \frac{h}{3} \{ (y_0 + y_6) + 4(y_1 + y_3 + y_5) \\ &\quad + 2(y_2 + y_4) \} \\ &= \frac{1}{18} \{ (1 + 1.4427) + 4(1.0812 \\ &\quad + 1.2332 + 1.3748) + 2(1.1587 \\ &\quad + 1.3051) \} \\ &\approx 1.2293. \end{aligned}$$

2539. 取 $n = 10$ ，计算加达郎常数

$$C = \int_0^1 \frac{\operatorname{arc} \operatorname{tg} x}{x} dx.$$

解 $h = \frac{1}{10}.$

$$x_0 = 0, \quad y_0 = 1;$$

$$x_1 = 0.1, \quad y_1 = 0.99669;$$

$$x_2 = 0.2, \quad y_2 = 0.98698;$$

$$x_3 = 0.3, \quad y_3 = 0.97152;$$

$$x_4 = 0.4, \quad y_4 = 0.95127;$$

$$x_5 = 0.5, \quad y_5 = 0.92730;$$

$$x_6 = 0.6, \quad y_6 = 0.90070;$$

$$x_7 = 0.7, \quad y_7 = 0.87247;$$

$$x_8 = 0.8, \quad y_8 = 0.84343;$$

$$x_9 = 0.9, \quad y_9 = 0.81424;$$

$$x_{10} = 1, \quad y_{10} = 0.78540.$$

按辛普森公式, 得

$$\begin{aligned} G &= \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) \\ &\quad + 2(y_2 + y_4 + y_6 + y_8)] \\ &= \frac{1}{30} (1.78540 + 18.32888 + 7.36476) \\ &= 0.91597. \end{aligned}$$

2540. 利用公式

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

计算数 π 精确到 10^{-6} .

解 利用辛普森公式计算其误差

$$R_n(x) = -\frac{(b-a)^6}{180n^4} f^{(4)}(\xi) \quad (a \leq \xi \leq b).$$

现在 $f(x) = \frac{1}{1+x^2}$, 事实上, 它是 $y = \arctg x$

的导数, 因而

$$f^{(4)}(x) = (\arctg x)^{(5)}.$$

利用第二章1218题的结果得知

$$f^{(4)}(x) = \frac{24}{(1+x^2)^{\frac{5}{2}}} \sin\left(5 \arctg \frac{1}{x}\right).$$

在区间 $[0, 1]$ 上,

$$|f^{(4)}(x)| \leq 24,$$

所以

$$|R_n(x)| \leq \frac{24}{180n^4}.$$

欲误差小于 0.00001 , 只需

$$\frac{24}{180n^4} \leq \frac{1}{100000},$$

即只需取 $n = 12$, 就有 $|R_n| \leq 6.5 \times 10^{-6}$.

其次, 我们还必须加进近似于函数值的误差, 设法使这个新的误差小于 3.6×10^{-6} , 这样, 就能保证总误差小于 10^{-5} . 为了这个目的, 只要计算 $\frac{1}{1+x^2}$ 的值到六位小数精确到 0.5×10^{-6} 就够了.

现取 $n = 12$, 则有

$$x_0 = 0, \quad y_0 = 1;$$

$$x_1 = \frac{1}{12}, \quad y_1 = 0.993103;$$

$$x_2 = \frac{1}{6}, \quad y_2 = 0.972973;$$

$$x_3 = \frac{1}{4}, \quad y_3 = 0.941176;$$

$$x_4 = \frac{1}{3}, \quad y_4 = 0.900000;$$

$$x_5 = \frac{5}{12}, \quad y_5 = 0.852071;$$

$$x_6 = \frac{1}{2}, y_6 = 0.800000;$$

$$x_7 = \frac{7}{12}, y_7 = 0.746114;$$

$$x_8 = \frac{2}{3}, y_8 = 0.692308;$$

$$x_9 = \frac{3}{4}, y_9 = 0.640000;$$

$$x_{10} = \frac{5}{6}, y_{10} = 0.590164;$$

$$x_{11} = \frac{11}{12}, y_{11} = 0.543396;$$

$$x_{12} = 1, y_{12} = 0.500000.$$

最后得到

$$\begin{aligned} \frac{\pi}{4} &= \int_0^1 \frac{dx}{1+x^2} \\ &= \frac{1}{36} [(y_0 + y_{12}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 \\ &\quad + y_{11}) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})] \\ &= 0.785398, \end{aligned}$$

所以

$$\pi = 0.785398 \times 4 = 3.14159,$$

精确到0.00001.

2541. 计算

$$\int_0^1 e^{x^2} dx$$

精确到0.001.

解 采用辛普森公式计算, 则其误差

$$R_n(x) = -\frac{1}{180n^4} 2e^{\xi^2} (8\xi^4 + 24\xi^2 + 6)$$
$$(0 < \xi < 1),$$

故有 $|R_n(x)| < \frac{1}{180n^4} \cdot 2e \cdot 38.$

要 $|R_n(x)| < 10^{-3}$, 只须 $\frac{2 \cdot 38 \cdot e^1}{180 n^4} < 10^{-3}$, 即只须取 $n = 6$.

现取 $n = 6$, 则有

$$x_0 = 0, \quad y_0 = 1;$$

$$x_1 = \frac{1}{6}, \quad y_1 = e^{\frac{1}{36}} = 1.0282;$$

$$x_2 = \frac{1}{3}, \quad y_2 = e^{\frac{1}{9}} = 1.1175;$$

$$x_3 = \frac{1}{2}, \quad y_3 = e^{\frac{1}{4}} = 1.2840;$$

$$x_4 = \frac{2}{3}, \quad y_4 = e^{\frac{4}{9}} = 1.5596;$$

$$x_5 = \frac{5}{6}, \quad y_5 = e^{\frac{25}{36}} = 2.0026;$$

$$x_6 = 1, \quad y_6 = e = 2.7183.$$

于是,

$$\int_0^1 e^{x^2} dx = \frac{1}{18} [(y_0 + y_6) + 4(y_1 + y_3 + y_5)]$$

$$+2(y_2 + y_4)] \doteq 1.463.$$

2542. 计算

$$\int_0^1 (e^x - 1) \ln \frac{1}{x} dx \text{ 精确到 } 10^{-4}.$$

解 对于函数 $f(x) = e^x$ 在 $0 \leq x \leq 1$ 上采用台劳展式以及相应的拉格朗日余项公式来估算误差:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \Delta_{n+1},$$

其中

$$\begin{aligned} \Delta_{n+1} &= \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} \\ &= \frac{e^{\theta x}}{(n+1)!} x^{n+1} \quad (0 < \theta < 1). \end{aligned}$$

于是,

$$|\Delta_{n+1}| \leq \frac{e}{(n+1)!} x^{n+1},$$

从而原来的积分数值为

$$\begin{aligned} I &= \int_0^1 (e^x - 1) \ln \frac{1}{x} dx \\ &= \sum_{k=1}^n \frac{1}{k!} \int_0^1 x^k \ln \frac{1}{x} dx + R_{n+1}, \end{aligned}$$

其中

$$|R_{n+1}| = \left| \int_0^1 \Delta_{n+1} \ln \frac{1}{x} dx \right|$$

$$\leq \frac{e}{(n+1)!} \int_0^1 x^{n+1} \ln \frac{1}{x} dx.$$

记 $I_k = \int_0^1 x^k \ln \frac{1}{x} dx$ ($k \geq 1$), 则有

$$\begin{aligned} I_k &= \frac{1}{k+1} \int_0^1 \ln \frac{1}{x} d(x^{k+1}) \\ &= \frac{1}{k+1} x^{k+1} \ln \frac{1}{x} \Big|_0^1 + \frac{1}{k+1} \int_0^1 x^k dx \\ &= \frac{1}{(k+1)^2}. \end{aligned}$$

如果取 $n = 5$, 则有

$$\begin{aligned} |R_6| &\leq \frac{e}{6!} I_6 = \frac{e}{6!} \cdot \frac{1}{7^2} = \frac{e}{7 \times 7!} \\ &= \frac{e}{35280} < \frac{3}{35280} < \frac{1}{1.1 \times 10^4} < 10^{-4}. \end{aligned}$$

记 $I = J + R_6$, 则有

$$\begin{aligned} J &= \sum_{k=1}^5 \frac{1}{k!} I_k = \sum_{k=1}^5 \frac{1}{k!} \cdot \frac{1}{(k+1)^2} \\ &= \sum_{k=1}^5 \frac{1}{(k+1)! (k+1)} \\ &= \frac{1}{2!2} + \frac{1}{3!3} + \frac{1}{4!4} + \frac{1}{5!5} + \frac{1}{6!6} \\ &= \frac{1}{4} + \frac{1}{18} + \frac{1}{96} + \frac{1}{600} + \frac{1}{4320} \\ &= 0.31787^+ = 0.3179 + \Delta', \end{aligned}$$

其中 $|\Delta'| \leq 0.00004 = 4 \times 10^{-5}$ 且 $\Delta' < 0$,

注意到由 $\Delta_{n+1} > 0$ 即可推知 $R_{n+1} > 0$, 于是

$$\begin{aligned} I &= J + R_6 = 0.3179 + (R_6 + \Delta') \\ &= 0.3179 + (R_6 - |\Delta'|) = 0.3179 + \Delta, \end{aligned}$$

且有 $I \doteq 0.3179$, 而此时其相应的误差已有

$$\begin{aligned} |\Delta| &= \left| R_6 - |\Delta'| \right| \leq \begin{cases} R_6, & \text{若 } |\Delta'| \leq R_6, \\ |\Delta'|, & \text{若 } |\Delta'| > R_6 \end{cases} \\ &\leq \max(R_6, |\Delta'|) < 10^{-4}. \end{aligned}$$

注 本题不能直接利用辛普森公式来计算所给的定积分的近似值, 因为被积函数 $(e^x - 1) \ln \frac{1}{x}$ 的四阶导函数在 $x=0$ 的右近旁是无界的, 从而不能估计出误差. 所以, 上面我们用台劳公式来作近似计算. 这样, 计算以及估计误差都较为简单. 当然, 也可间接地利用辛普森公式来计算所给定积分的近似值, 这时需要或者改变被积函数或者把积分区间分成两个. 例如, 我们可以改变被积函数如下: 令

$$I = \int_0^1 (e^x - 1) \ln \frac{1}{x} dx = - \int_0^1 (e^x - 1) \ln x dx,$$

设 $f(x) = (e^x - 1) \ln x$, 若补充定义

$$f(0) = \lim_{x \rightarrow +0} f(x) = 0,$$

则 $f(x)$ 是 $0 \leq x \leq 1$ 上的连续函数. 由于

$$\begin{aligned} f'(x) &= e^x \ln x + \frac{e^x - 1}{x} \\ &= f(x) + \frac{e^x - 1}{x} + \ln x \quad (0 < x \leq 1), \end{aligned}$$

故

$$\int_0^1 f'(x) dx = \int_0^1 f(x) dx + \int_0^1 \frac{e^x - 1}{x} dx + \int_0^1 \ln x dx.$$

注意到

$$\int_0^1 f'(x) dx = f(1) - f(0) = 0,$$

$$\int_0^1 \ln x dx = (x \ln x - x) \Big|_0^1 = -1,$$

得

$$I = \int_0^1 \frac{e^x - 1}{x} dx - 1.$$

于是，我们把求 $\int_0^1 (e^x - 1) \ln \frac{1}{x} dx$ 的近似值问题，归

结为求 $\int_0^1 \frac{e^x - 1}{x} dx$ 的近似值问题。令 $g(x) = \frac{e^x - 1}{x}$,

并补充定义

$$g(0) = \lim_{x \rightarrow 0} g(x) = 1,$$

则 $g(x)$ 是 $0 \leq x \leq 1$ 上的连续函数。由求高阶导数的莱布尼兹法则，易得

$$g^{(n)}(x) = \frac{e^x P_n(x) - (-1)^n n!}{x^{n+1}} \quad (0 < x \leq 1),$$

其中 $P_n(x) = \sum_{k=0}^n C_n^k (-1)^k k! x^{n-k} \quad (n=1, 2, \dots)$.

下面证明 $g^{(n)}(0)$ 存在并且 $g^{(n)}(0) = -\frac{1}{n+1}$ ($n = 1, 2, \dots$). 首先, 由洛比塔法则, 我们有

$$\begin{aligned} \lim_{x \rightarrow +0} g^{(n)}(x) &= \lim_{x \rightarrow +0} \frac{e^x P_n(x) - (-1)^n n!}{x^{n+1}} \\ &= \lim_{x \rightarrow +0} \frac{e^x [P_n(x) + P'_n(x)]}{(n+1)x^n} \\ &= \lim_{x \rightarrow +0} \frac{e^x x^n}{(n+1)x^n} = \frac{1}{n+1} \quad (n=1, 2, \dots). \end{aligned}$$

于是, 根据中值定理, 得

$$\begin{aligned} g'(0) &= \lim_{x \rightarrow +0} \frac{g(x) - g(0)}{x - 0} = \lim_{\xi \rightarrow +0} g'(\xi) \\ &= \frac{1}{2} \quad (0 < \xi < x). \end{aligned}$$

今假定 $g^{(n)}(0)$ 存在且 $g^{(n)}(0) = \frac{1}{n+1}$. 于是,

$$\begin{aligned} g^{(n+1)}(0) &= \lim_{x \rightarrow +0} \frac{g^{(n)}(x) - g^{(n)}(0)}{x - 0} \\ &= \lim_{\eta \rightarrow +0} g^{(n+1)}(\eta) = \frac{1}{n+2} \quad (0 < \eta < x). \end{aligned}$$

根据数学归纳法, 知 $g^{(n)}(0)$ 存在且

$$g^{(n)}(0) = \frac{1}{n+1} \quad (n=1, 2, \dots).$$

由此又知 $g^{(n)}(x)$ 是 $0 \leq x \leq 1$ 上的连续函数 ($n=1, 2, \dots$)。令 $h(x) = e^x P_n(x) - (-1)^n n!$ 。由于

$$h'(x) = e^x [P_n(x) + P_n'(x)] = e^x x^n > 0$$

(当 $0 < x \leq 1$ 时),

故 $h(x)$ 在 $[0, 1]$ 上是严格增大的, 从而

$$h(x) > h(0) = 0 \quad (\text{当 } 0 < x \leq 1 \text{ 时}).$$

因此, 当 $0 < x \leq 1$ 时 $g^{(n)}(x) > 0$ ($n=1, 2, \dots$), 所以 $g^{(n-1)}(x)$ 是 $0 \leq x \leq 1$ 上的严格增函数 ($n=1, 2, \dots$)。特别, $g^{(4)}(x)$ 当然是 $0 \leq x \leq 1$ 上的严格增函数。于是, 当 $0 \leq x \leq 1$ 时, 恒有

$$\frac{1}{5} = g^{(4)}(0) \leq g^{(4)}(x) \leq g^{(4)}(1).$$

由于当 $0 < x \leq 1$ 时,

$$g^{(4)}(x) = \frac{e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) - 24}{x^5},$$

故 $g^{(4)}(1) = 9e - 24 < 0.5$ 。因此, 当 $0 \leq x \leq 1$ 时,

$$0.2 \leq g^{(4)}(x) \leq 0.5.$$

代入辛普森公式的误差表达式, 得

$$|R_n(x)| = \left| -\frac{g^{(4)}(\xi'')}{180 \cdot n^4} \right| \leq \frac{1}{360n^4},$$

$$R_n(x) < 0.$$

取 $n=4$, 有

$$|R_4(x)| \leq \frac{1}{360n^4} < 1.1 \times 10^{-5}.$$

计算得

$$g(0)=1, \quad g\left(\frac{1}{4}\right)=1.13610, \quad g\left(\frac{1}{2}\right)=1.29744,$$

$$g\left(\frac{3}{4}\right)=1.48933, \quad g(1)=1.71828.$$

于是，代入后最终得

$$\begin{aligned} I &= \int_0^1 g(x) dx - 1 \doteq \frac{1}{12} \left\{ g(0) + g(1) + 2g\left(\frac{1}{2}\right) \right. \\ &\quad \left. + 4 \left[g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) \right] \right\} - 1 \\ &= 1.3179 - 1 = 0.3179, \end{aligned}$$

其误差的绝对值显然小于 $0.0001 = 10^{-4}$ 。

也可不改变被积函数，而把积分区间分成两个，步骤如下：

$$\text{令 } u = \frac{1-x}{x} \quad (0 < x < 1), \quad \text{则 } \frac{1}{x} = 1 + u \quad (u > 0).$$

于是，当 $0 < x < 1$ 时，有

$$\begin{aligned} 0 &< (e^x - 1) \ln \frac{1}{x} = (e^x - 1) \ln(1 + u) \\ &< (e^x - 1) u = \frac{1-x}{x} (e^x - 1) < \frac{e^x - 1}{x}. \end{aligned}$$

前面已证函数 $g(x) = \frac{e^x - 1}{x}$ 在 $0 \leq x \leq 1$ 上是严格增

大的（注意，规定 $g(0) = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ ），故当

$0 < x < 1$ 时，有

$$1 < \frac{e^x - 1}{x} < g(1) = e - 1 < 2;$$

从而

$$\begin{aligned} 0 &< \int_0^{10^{-5}} (e^x - 1) \ln \frac{1}{x} dx < \int_0^{10^{-5}} \frac{e^x - 1}{x} dx \\ &< 2 \int_0^{10^{-5}} dx = 0.2 \times 10^{-4}. \end{aligned}$$

求出函数 $(e^x - 1) \ln \frac{1}{x}$ 的四阶导函数的表达式后，易知它在闭区间 $10^{-5} \leq x \leq 1$ 上是连续的，从而是有界的，并且不难估计出其绝对值的上界。因此，可利用辛普森公式计算积分

$$\int_{10^{-5}}^1 (e^x - 1) \ln \frac{1}{x} dx$$

的近似值，使误差的绝对值小于 0.8×10^{-4} 。显然，若以此作为积分 $\int_0^1 (e^x - 1) \ln \frac{1}{x}$ 的近似值，则其误差的绝对值小于 10^{-4} 。由于计算较繁，从略。

2543. 近似地计算概率积分

$$\int_0^{+\infty} e^{-x^2} dx.$$

解 作变换

$$x = \frac{t}{1-t},$$

则积分

$$\int_0^{+\infty} e^{-x^2} dx = \int_0^1 e^{-\left(\frac{t}{1-t}\right)^2} \frac{1}{(1-t)^2} dt.$$

由于题中对精确度未提出明确要求,故 n 可任取.例如

取 $n=2k=18$, $\Delta t = \frac{1}{18}$, 则有

$$t_0 = 0, \quad y_0 = 1;$$

$$t_1 = \frac{1}{18}, \quad 4y_1 = 4.46894;$$

$$t_2 = \frac{2}{9}, \quad 2y_2 = 2.49201;$$

$$t_3 = \frac{1}{6}, \quad 4y_3 = 5.53415;$$

$$t_4 = \frac{2}{9}, \quad 2y_4 = 3.04696;$$

$$t_5 = \frac{5}{18}, \quad 4y_5 = 6.61414;$$

$$t_6 = \frac{1}{3}, \quad 2y_6 = 3.50460;$$

$$t_7 = \frac{7}{18}, \quad 4y_7 = 7.14411;$$

$$t_8 = \frac{4}{9}, \quad 2y_8 = 3.41685;$$

$$t_9 = \frac{1}{2}, \quad 4y_9 = 5.88607;$$

$$t_{10} = \frac{5}{9}, \quad 2y_{10} = 2.12232;$$

$$t_{11} = \frac{11}{18}, \quad 4y_{11} = 2.23855;$$

$$t_{12} = \frac{2}{3}, \quad 2y_{12} = 0.32968;$$

$$t_{13} = \frac{13}{18}, \quad 4y_{13} = 0.06009;$$

$$t_{14} = \frac{7}{9}, \quad 2y_{14} = 0.00010;$$

$$t_{15} = \frac{5}{6}, \quad 4y_{15} = 0;$$

$$t_{16} = \frac{8}{9}, \quad 2y_{16} = 0;$$

$$t_{17} = \frac{17}{18}, \quad 4y_{17} = 0;$$

$$t_{18} = 1, \quad y_{18} = \lim_{t \rightarrow 1} e^{-\left(\frac{t}{1-t}\right)^2} \left(\frac{1}{1-t}\right)^2 = 0.$$

按辛普森公式，得

$$\begin{aligned} \int_0^{+\infty} e^{-x^2} dx &= \int_0^1 e^{-\left(\frac{t}{1-t}\right)^2} \frac{1}{(1-t)^2} dt \\ &= \frac{1}{54} (1 + 4.46894 + 2.49201 \\ &\quad + 5.53415 + 3.04696 + 6.61414 \\ &\quad + 3.50460 + 7.14411 + 3.41685 \\ &\quad + 5.88607 + 2.12232 + 2.23855 \end{aligned}$$

$$+ 0.32968 + 0.06009 + 0.00010)$$

$$= \frac{47.85857}{54} \doteq 0.88627.$$

2544. 近似地求出半轴为 $a=10$ 及 $b=6$ 的椭圆的周长.

解 设椭圆的参数方程为

$$x = 10 \cos t, \quad y = 6 \sin t.$$

$$\text{于是有 } ds = \sqrt{x'^2 + y'^2} dt = 10 \sqrt{1 - \frac{16}{25} \sin^2 t} dt,$$

从而得椭圆的周长为

$$s = 4 \int_0^{\frac{\pi}{2}} ds = 40 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25} \sin^2 t} dt.$$

现取 $n=2k=6$ 近似计算积分

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25} \sin^2 t} dt.$$

$$\text{注意到 } \sin^2 \frac{\pi}{12} = \frac{2 - \sqrt{3}}{4}, \quad \sin^2 \frac{5\pi}{12} = \frac{2 + \sqrt{3}}{4},$$

$$h = \frac{\pi}{12}, \text{ 即有}$$

$$t_0 = 0, \quad y_0 = 1;$$

$$t_1 = \frac{\pi}{12}, \quad 4y_1 = 4 \sqrt{1 - \frac{16}{25} \cdot \frac{1}{4} (2 - \sqrt{3})}$$

$$= 3.913;$$

$$t_2 = \frac{\pi}{6}, \quad 2y_2 = 2 \sqrt{1 - \frac{16}{25} \cdot \frac{1}{4}} = 1.833;$$

$$t_3 = \frac{\pi}{4}, \quad 4y_3 = 4\sqrt{1 - \frac{16}{25} \cdot \frac{1}{2}} = 3.293,$$

$$t_4 = \frac{\pi}{3}, \quad 2y_4 = 2\sqrt{1 - \frac{16}{25} \cdot \frac{3}{4}} = 1.442,$$

$$t_5 = \frac{5\pi}{12}, \quad 4y_5 = 4\sqrt{1 - \frac{16}{25} \cdot \frac{1}{4}(2 + \sqrt{3})}$$

$$= 2.539;$$

$$t_6 = \frac{\pi}{2}, \quad y_6 = \sqrt{1 - \frac{16}{25}} = 0.6.$$

按辛普森公式，得

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25} \sin^2 t} dt$$

$$= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{36} (1 + 0.6 + 3.913 + 3.298 + 2.539 + 1.833$$

$$+ 1.442)$$

$$\approx 1.276,$$

所以，椭圆周长的近似值为

$$s = 40 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{16}{25} \sin^2 t} dt$$

$$= 40 \times 1.276 = 51.04.$$

2545. 取 $\Delta x = \frac{\pi}{3}$ ，按点子作出函数

$$y = \int_0^x \frac{\sin t}{t} dt \quad (0 \leq x \leq 2\pi)$$

的图形。

解 取 $n=2k=6$ 计算函数 $y = \int_0^x \frac{\sin t}{t} dt$ 的值。

先计算 $y = \int_0^{\frac{\pi}{3}} \frac{\sin t}{t} dt$ 。由于 $h = \frac{\pi}{18}$ ，且

$$t_0 = 0, \quad y_0 = 1;$$

$$t_1 = \frac{\pi}{18}, \quad 4y_1 = 3.980;$$

$$t_2 = \frac{\pi}{9}, \quad 2y_2 = 1.960;$$

$$t_3 = \frac{\pi}{6}, \quad 4y_3 = 3.820;$$

$$t_4 = \frac{2\pi}{9}, \quad 2y_4 = 1.841;$$

$$t_5 = \frac{5\pi}{18}, \quad 4y_5 = 3.511;$$

$$t_6 = \frac{\pi}{3}, \quad y_6 = 0.827.$$

按辛普森公式，得

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{54} (1 + 0.827 + 3.980 + 3.820 \\ &\quad + 3.511 + 1.960 + 1.841) \\ &\doteq 0.99. \end{aligned}$$

再计算 $y = \int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt$. 由于 $h = \frac{\pi}{9}$, 且

$$t_0 = 0, \quad y_0 = 1;$$

$$t_1 = \frac{\pi}{9}, \quad 4y_1 = 3.919;$$

$$t_2 = \frac{2\pi}{9}, \quad 2y_2 = 1.841;$$

$$t_3 = \frac{\pi}{3}, \quad 4y_3 = 3.308;$$

$$t_4 = \frac{4\pi}{9}, \quad 2y_4 = 1.411;$$

$$t_5 = \frac{5\pi}{9}, \quad 4y_5 = 2.257;$$

$$t_6 = \frac{2\pi}{3}, \quad y_6 = 0.413.$$

所以,

$$\begin{aligned} \int_0^{\frac{2\pi}{3}} \frac{\sin t}{t} dt &= \frac{\pi}{27} (1 + 0.413 + 3.919 + 3.308 \\ &\quad + 2.257 + 1.841 + 1.411) \\ &\doteq 1.65. \end{aligned}$$

选取适当的 n , 类似地可求得

$$\int_0^{\pi} \frac{\sin t}{t} dt \doteq 1.85; \quad \int_0^{\frac{4\pi}{3}} \frac{\sin t}{t} dt \doteq 1.72;$$

$$\int_0^{\frac{5\pi}{3}} \frac{\sin t}{t} dt \doteq 1.52; \quad \int_0^{2\pi} \frac{\sin t}{t} dt \doteq 1.42.$$

列表作图如下（图4.55）：

| | | | | | | | |
|-----|---|-----------------|------------------|-------|------------------|------------------|--------|
| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | 2π |
| y | 0 | 0.99 | 1.65 | 1.95 | 1.72 | 1.52 | 1.42 |

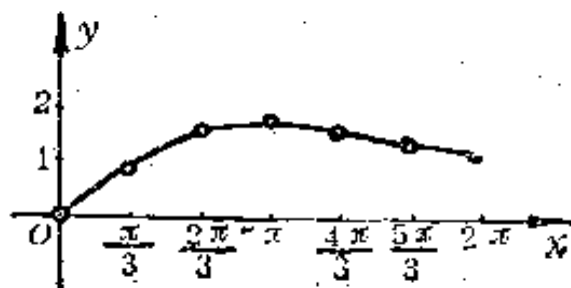


图 4.55